Periodic splitting sequences of the twice punctured torus

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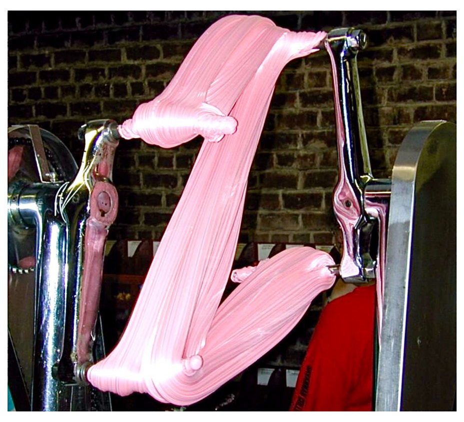
Motivation

Many real-world phenomena can be modelled by surface maps, i.e. maps that move points on a two-dimensional surface. Typical examples are a three-rod stirring device, a dough-mixer or a taffy puller. Under the movement of the rods, the particles are moved around. The movement of the particles can be described by a map and studied mathematically. For example, we can calculate whether a given map mixes well and calculate a number, describing the mixing efficiency. Especially so-called *pseudo-Anosov maps* are known to stir well and uniformly. I investigated some „mixing maps“ on the twice-punctured torus which are known to be pseudo-Anosov. I describe how those maps mix points on the surface and use train tracks to understand their properties.



Background

We will study a torus with two points removed. In the following, we draw the torus as a rectangle. The rectangle can be obtained by cutting open the torus along two curves, as seen in figure 1. We can „mix“ points on the torus by applying right of left-handed *Dehn twists*. The right-handed Dehn Twist along a curve is illustrated in figure 2. It cuts the torus open along a curve and twists one end counter-clockwise by a $360°$ rotation. Figure 3 shows right-handed Dehn twists along the curves c_1, c_2 and a left-handed Dehn-twist along c_3. We denote them by δ_1, δ_2, δ_3^{-1} respectively.

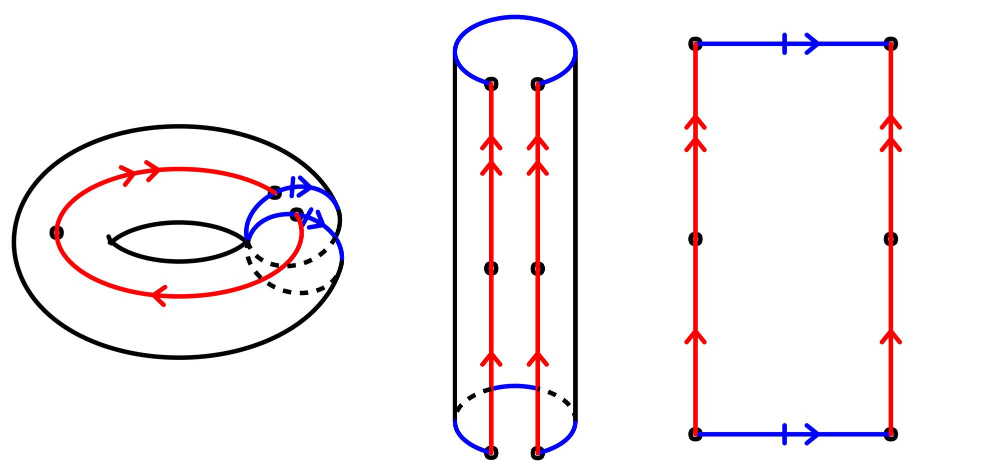


Fig. 1: The twice punctured torus cut open

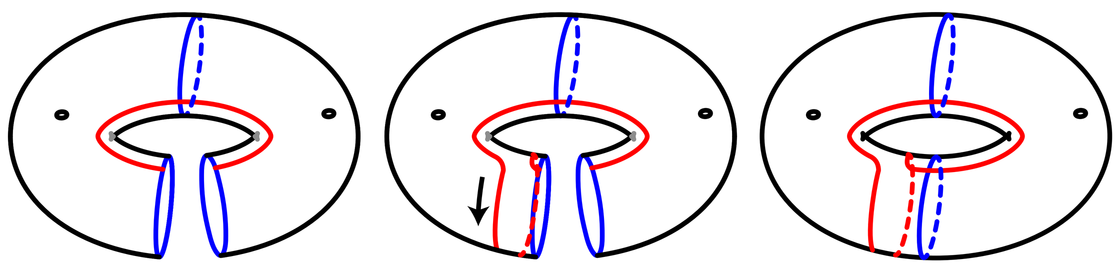


Fig. 2: Right-handed Dehn twist along the curve

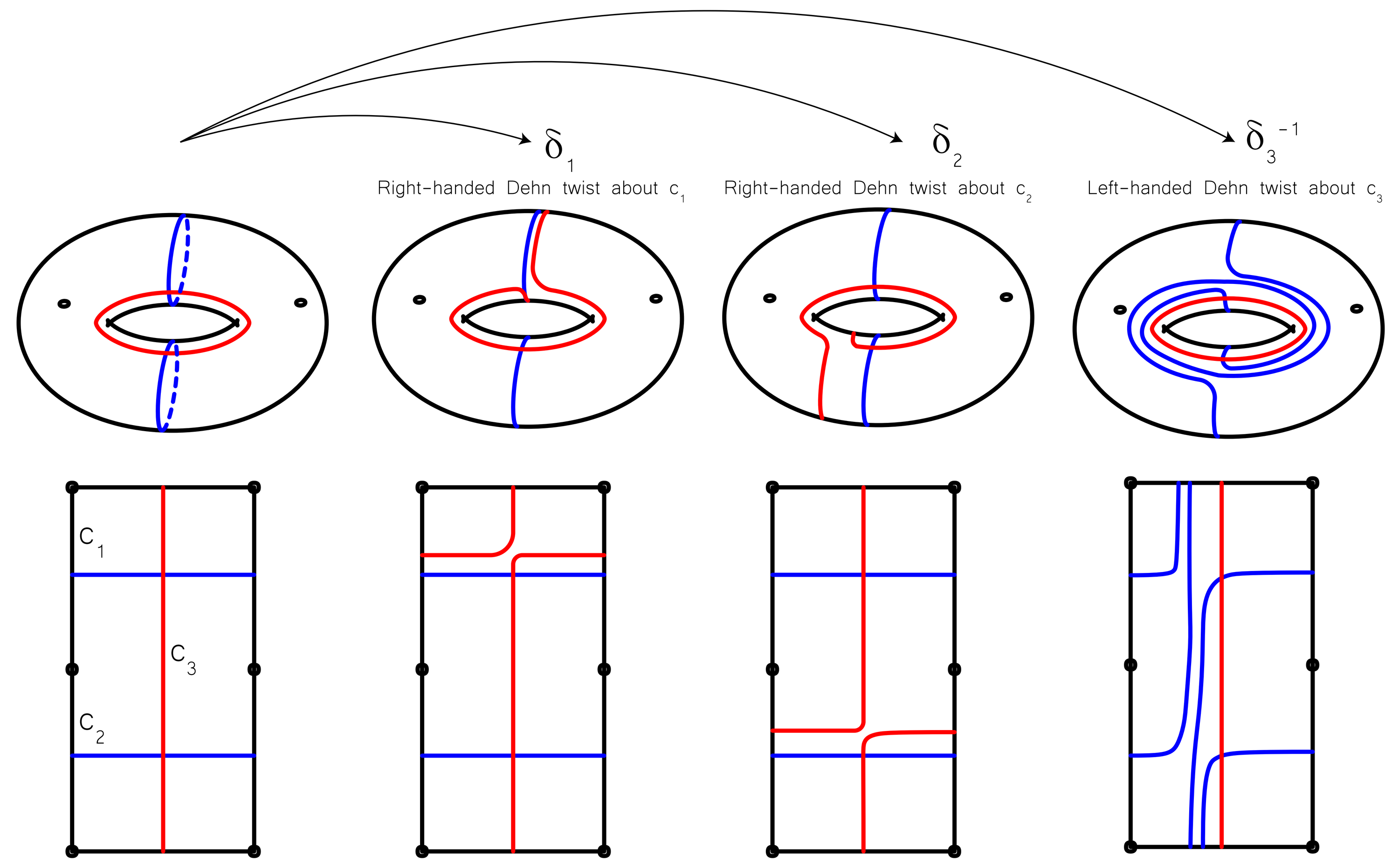


Fig. 3: The Dehn twist on the rectangle
Intersecting curves turn right after applying a right-handed twist

Foliations

We draw a curve on the torus (figure 4). Applying the map $f = \delta_3^{-1}\delta_2\delta_1$ repeatedly will stretch the curve over the torus. After a while, a *foliation*, consisting of *leaves* will appear. Applying f to the foliation will make the foliation denser but it won't change its slope. We partition the foliation into different segments. Then, we bundle leaves of the same colour into *branches* and assign the thickness of the segments as *weights* to the branches. The result is called a *train track* (figure 5). Every time two branches of the train track merge into one, the weights add up. Applying f to the train track can teach us how parts of the torus are folded under the mixing of f. (figure 6)

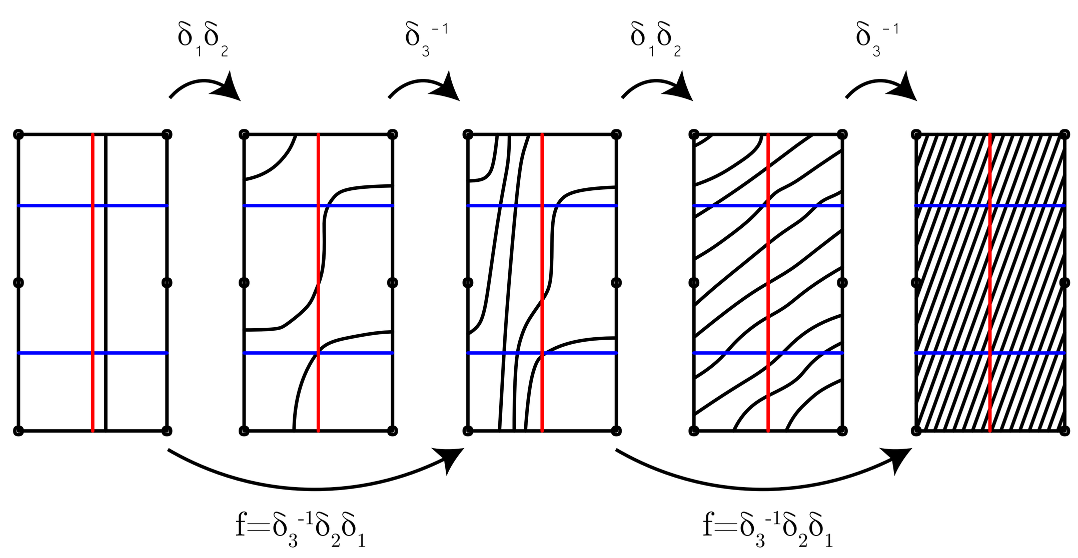


Fig. 4: Repeatedly applying the map f to a curve will make it into a foliation

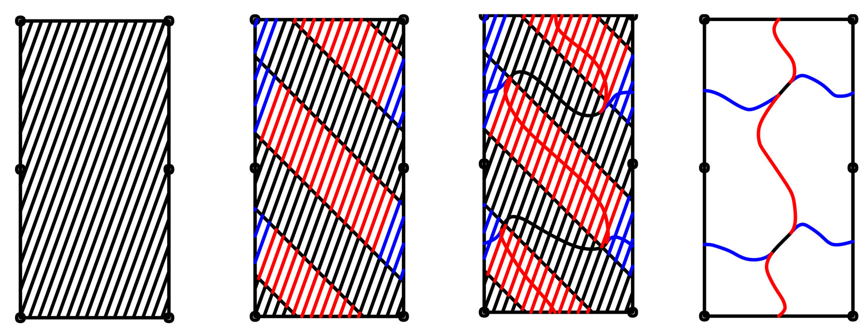


Fig. 5: Transforming the foliation into a train track

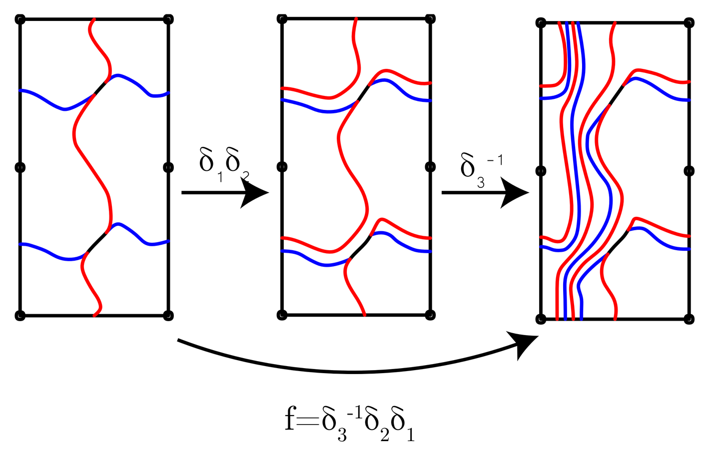


Fig. 6: We apply f to the train track

Maximal splitting

Using a *split*, we can modify a train track. For this, we split one branch into two. Then we connect the branches by a middle branch such that the weights of converging branches add up. Depending on the branch weights, this can be a left split or a right split. A *maximal split* splits all branches of a train track with the largest weight simultaneously. It is denoted by ⇀. Figure 8 shows a maximal split.

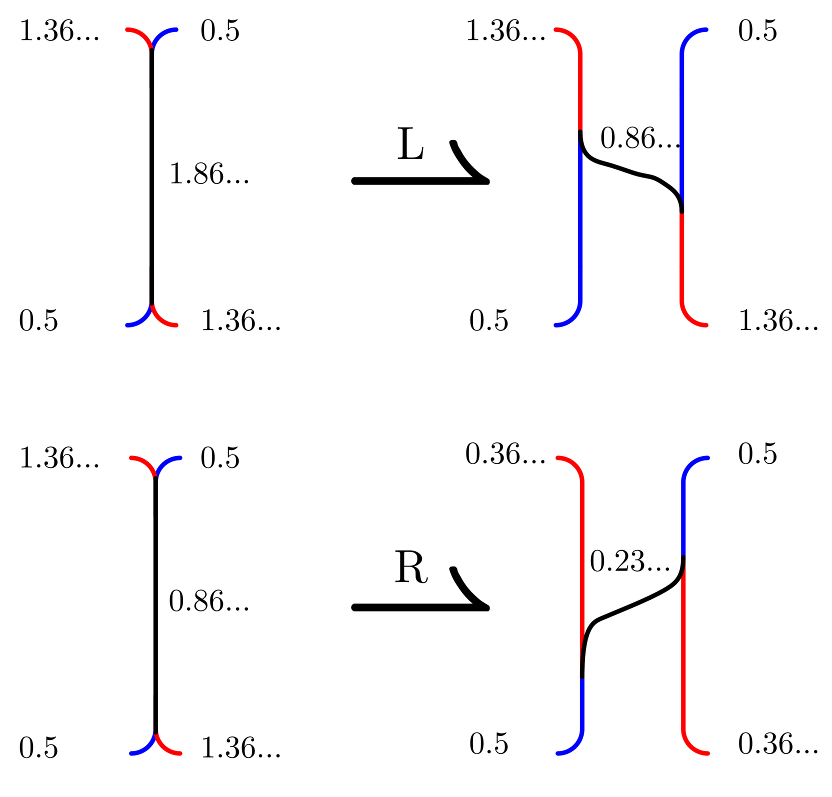
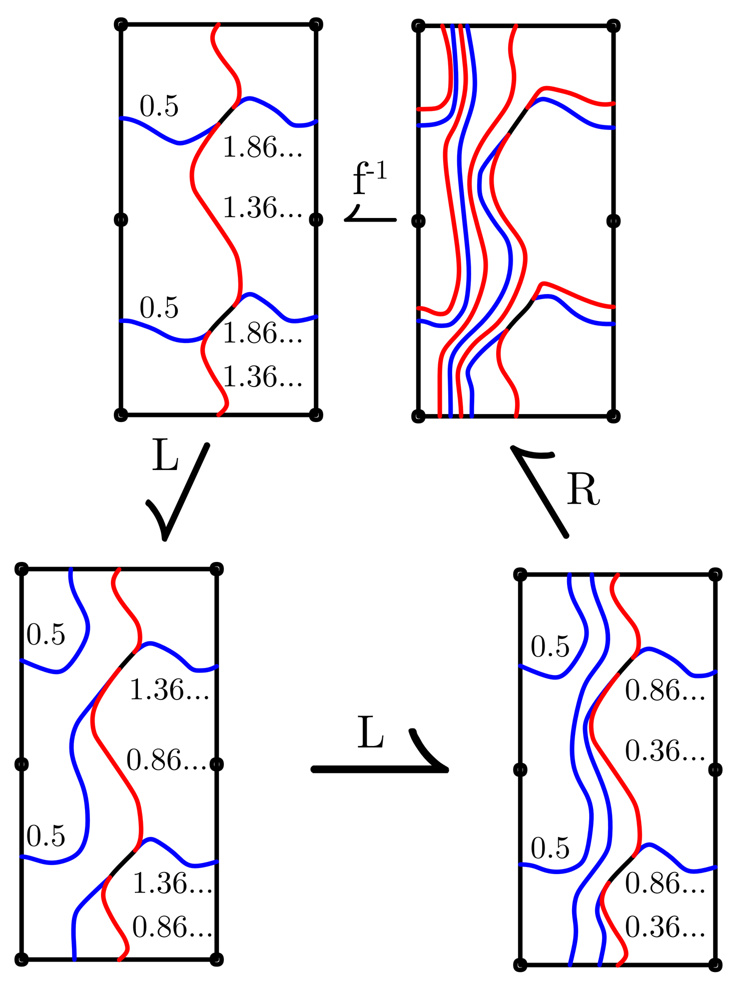


Fig. 7: A left and right split of a branch

Splitting Sequence

The repetition of maximal splits is called a *track splitting sequence*. When we repeat maximal splits on a train track, the train track gets more and more complicated. Agol showed 2011 that this sequence will eventually recreate the action of f (just as seen in the picture). This is useful, since it allows us to study f using maximal splits. However, it is still unknown how many maximal splits it takes to recreate the action or which train tracks lie inside the splitting sequence. In my research I explicitly calculated the cycle for a family of maps generalizing f.



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