Cayley Graphs

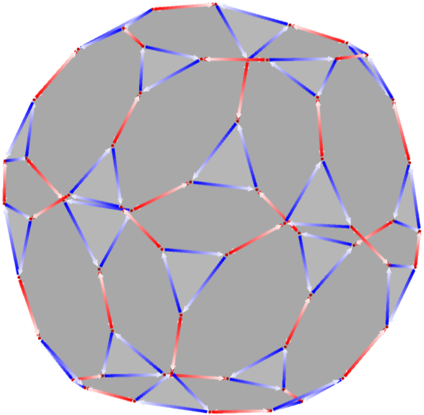

Figure: Symmetry group if the icosahedron I

Cayley Graphs

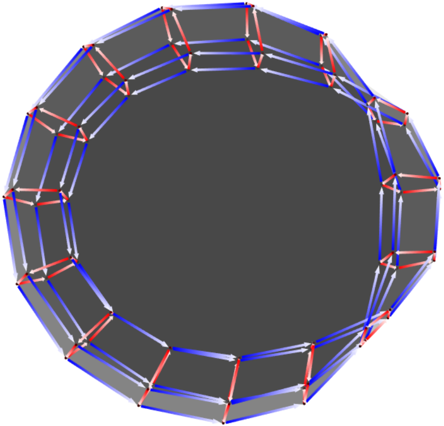

Figure: Product group $C_5 \times C_{15}$

Cayley Graphs

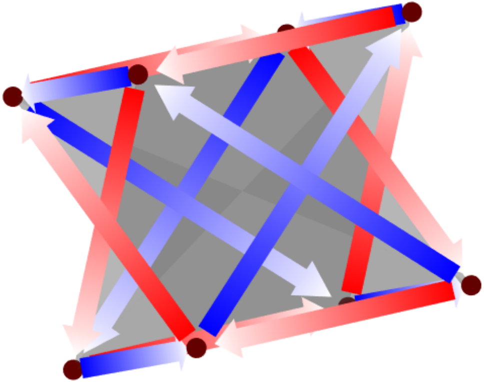

Figure: Quaternion group Q_8

Cayley Graphs

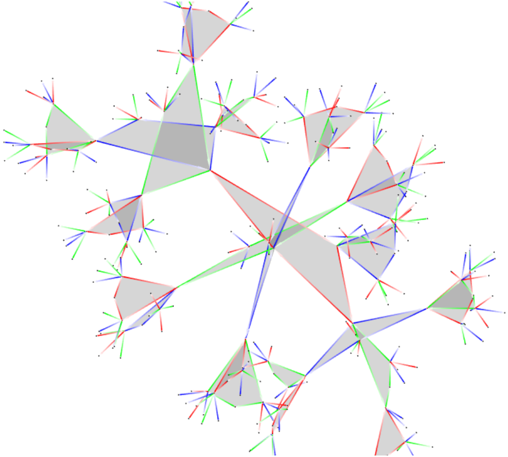

Figure: The group of the 2x2x2 Rubik's Cube

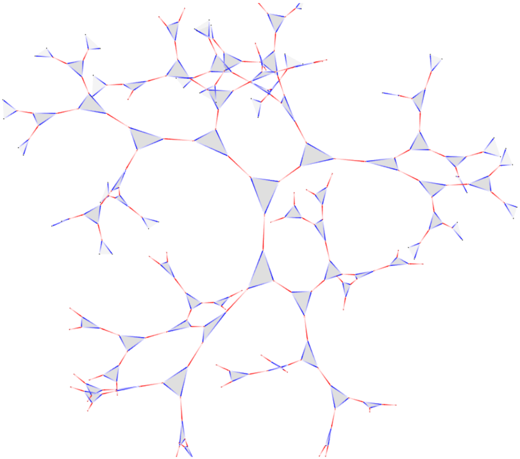

Figure: The group $PSL(2, \mathbb{Z})$

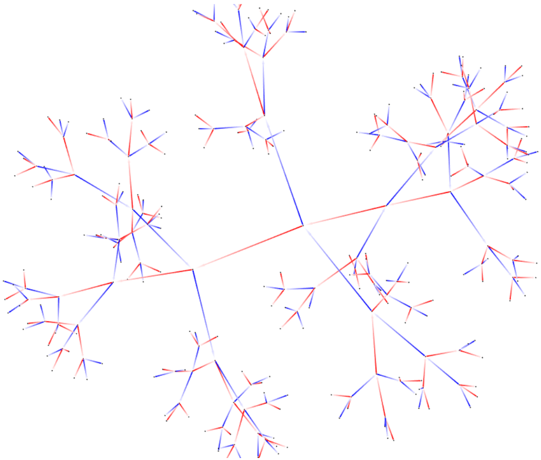

Figure: The free group F_2

Cayley Graphs

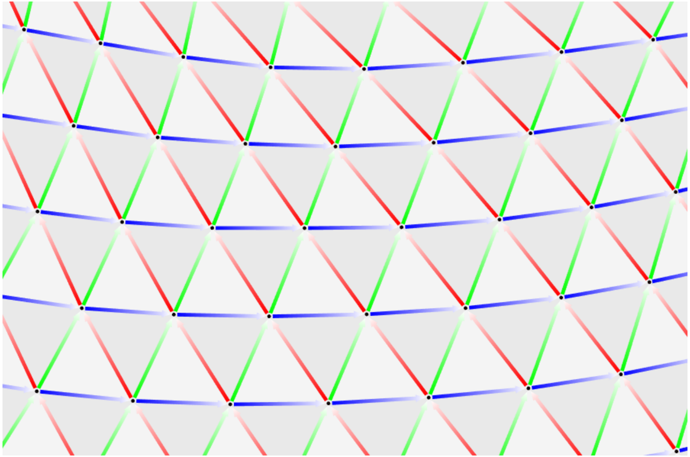

Figure: The group $\mathbb{Z}^2$

Cayley Graphs!

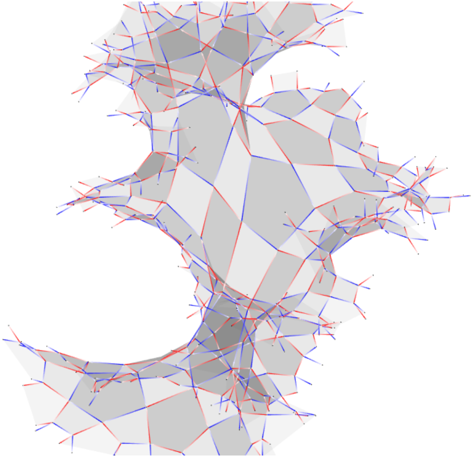

Figure: A simple hyperbolic group

Cayley Graphs!!!

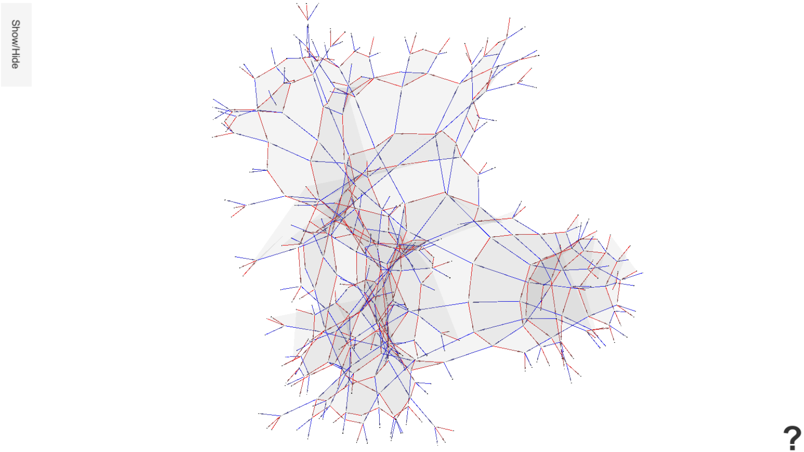

Figure: The Baumslag-Solitar group $BS(2,1)$

Cayley Graphs!!!!!!!!

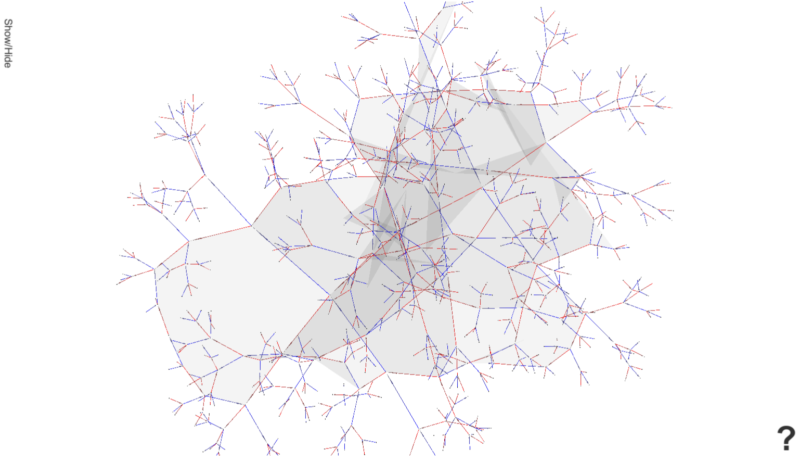

Figure: The Thompson group T

Visualizing Cayley graphs of finitely presented groups using force-directed graph algorithms

Jean-Baptiste Bellynck

Ludwig Maximilian University of Munich
Osaka University

19th East Asian Conference on Geometric Topology
19.02.2024

Objectives

The goals of this talk are to

- inform you about the application
- convince you to try the visualizations during this conference
- gather feedback and opinions from teaching and research experts
- make my first big step into academia

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What is the Cayley Graph Generator?
An open-source web-based application to generate Cayley graphs of finitely presented groups, written in cooperation with my co-student Johannes Heißler.

Johannes Heißler

Background I

Definition (Presentation)

A presentation of a group describes the group by a set of generating elements and their relationship to one another. It can be understood as an upgrade to a generating set of a group where we forget the ambient group and describe relationships using equalities.

Generating set:

$$\mathbb{Z}/6\mathbb{Z} \cong \langle 2 \rangle \subseteq \mathbb{Z}/12\mathbb{Z}$$

$\mathbb{Z}/6\mathbb{Z}$ generated using a generating set sitting inside $\mathbb{Z}/12\mathbb{Z}$.

Presentation:

$$\mathbb{Z}/6\mathbb{Z} \cong \langle a | a^6 = e \rangle$$

$\mathbb{Z}/6\mathbb{Z}$ generated using a presentation by specifying a relationship for the generator a.

Background II

Example

The presentation $\langle a, b | aba^{-1}b^{-1} \rangle$ describes the additive group $\mathbb{Z}^2$. The generators can be understood as the basis vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the relator can be understood as the following equality:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Definition (Finitely presented group)

A group is called finitely presented if it has a presentation with finitely many generators and relators.

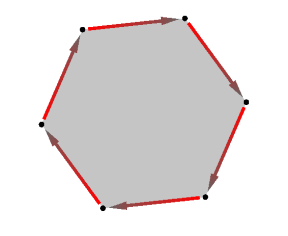
Background III

Definition (Cayley graph)

Let $G \cong \langle S|R \rangle$ be a presentation of a group G. The Cayley graph $\Gamma = (G, E)$ is a graph, where the verticies are the group elements G. Between two elements g, h there is a directed edge if and only if they are connected by right-multiplication of a generator $s \in S$, i.e. $h = g \cdot s$.

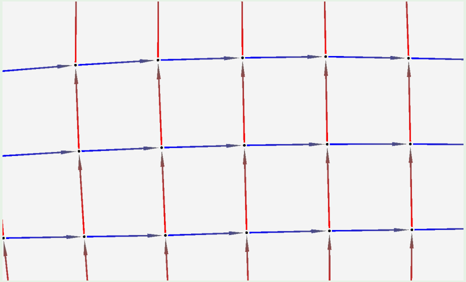
Example

The presentation $C_6 = \langle a|aaaaaa \rangle$ gives the following Cayley graph:



Example

The presentation $\mathbb{Z}^2 = \langle a, b | aba^{-1}b^{-1} \rangle$ gives the following Cayley graph:



Relators appear as cycles inside the graph. No matter at what vertex you begin, following the symbols of a relator will bring you back to your original group element.

Background V

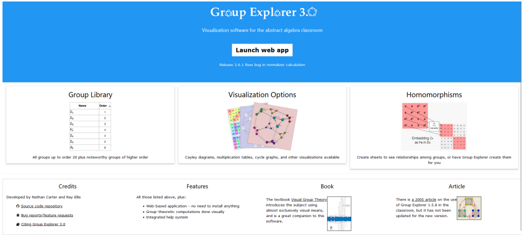
Definition (Cayley complex)

A Cayley complex can be created from a Cayley graph by adding a in face inside the cycles defined by the relator.

Previous work

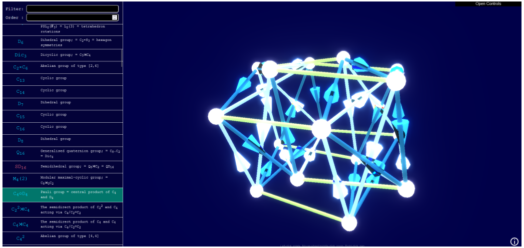
Similar applications have been developed, but they only allow for visualizations of finite groups up to order 500 and do not allow the input of a presentation.

Group Explorer by Nathan Carter



https://nathancarter.github.io/group-explorer/index.html

Cayley Graph Plotter by JuliaPoo



https://juliapoo.github.io/Cayley-Graph-Plotting/

Implementation

How to generate Cayley diagrams?

1. We place the identity element
2. For each generator and each generator inverse we draw an arrow with a new group element connected to the other end
3. For each drawn vertex we follow the symbols inside the relators
4. If we reach another vertex we merge the two vertices
5. If the merged vertex has multiple edges of the same generator attached the edges and their attached vertices are merged recursively.

Generating the graph of $\langle a, b | a^{-1} b^{-1} a b \rangle$

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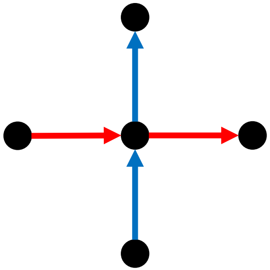

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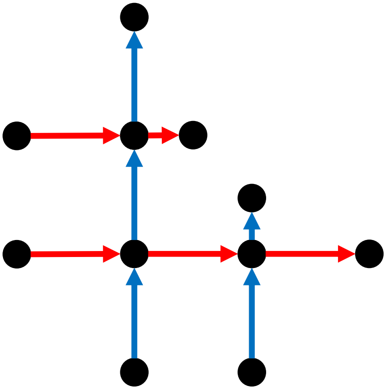

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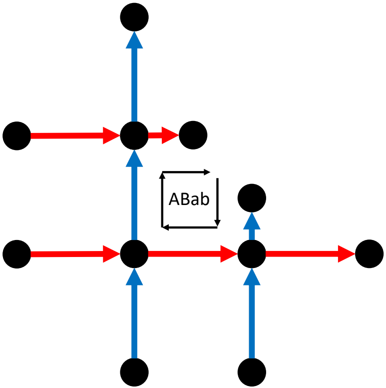

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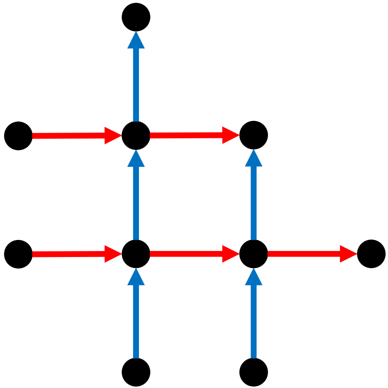

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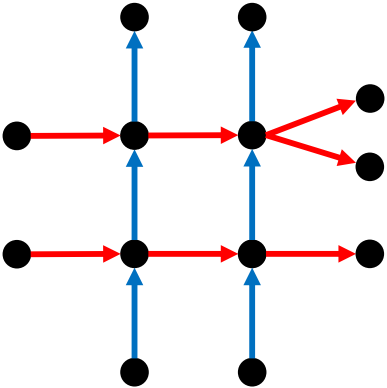

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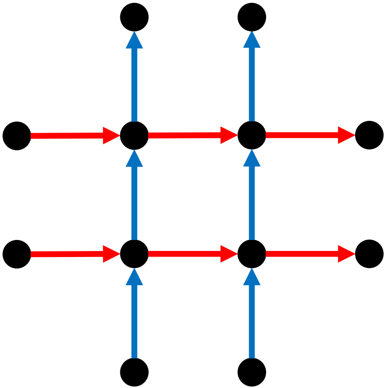
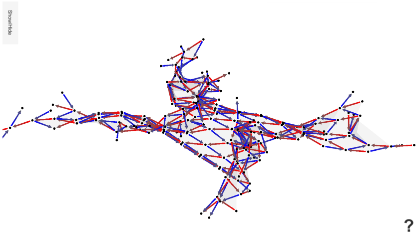
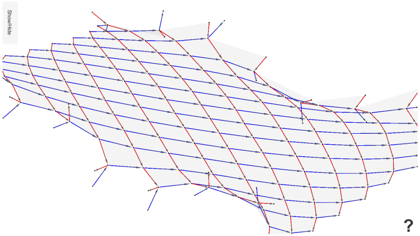


Generating the graph of $\langle a, b | a^{-1} b^{-1} ab \rangle$

How to make the diagrams look nice? When drawing a Cayley diagram we wish a graph which respects the symmetry of the group. This is realized by using a force-directed graph algorithm.



Not a pretty graph



A pretty graph

Definition

A force-directed graph algorithm models the vertices as charged particles repelling each other, the edges as springs attracting each other and then runs a physical simulation on the graph.

Demonstration

Feel free to join me on my live showcase. The application can be accessed from the smartphone using the QR-Code or by following the link: https://bit.ly/cayley

Please try it out! Try to find ways how you might use this in your own classroom or research and tell me at the end!

Uses in undergraduate education

The application gives us more ways to explain groups to undergraduate students:

- Characterize groups as a vertices of a Cayley graph
- Characterize groups as a set of operations on a Cayley graph
- Characterize generated subgroups as a subset of vertices
- Characterize cosets as the other subsets
- Probably much more

Uses in graduate education

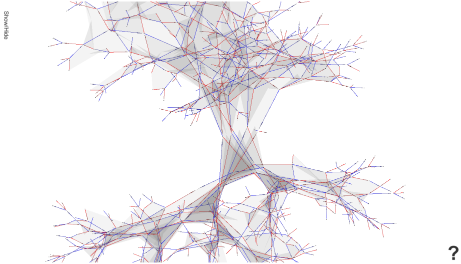
The application gives us novel ways to explain groups to graduate students:

- Give an understanding of group presentations
- Visualize operations on presentations (e.g. Tietze transformations, free product, amalgamated product, HNN-extentions)
- Visualize large-scale properties of groups (e.g. tree-like groups, groups with positive or negative curvature)

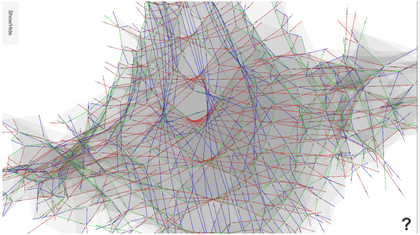
Uses in research

In research the application might be used in the following ways:

- An additional ways to understand an object from another perspective
- Explain collegues certain concepts
- Might lead to discoveries which are hard to see in algebraic form.



The braid group B_3 is tree-like in one directions and plane-like in another



The Heisenberg group H has ruled surfaces contained in it

How to contribute?

- Do you have any ideas for application in undergraduate, graduate education or research?
- Do you wish to see a feature or group implemented?
- Do you have any comments?

The application is a collaborative open-source project and can be found on GitHub.
(https://github.com/jeanbellynck/Cayley-Graph-Generator)
If you like to contribute, you can do it as follows:

- Tell me about it at this conference or write me an e-mail at j.bellynck@campus.lmu.de
- Create a issue/feature request on the GitHub page
- Implement the changes yourself and create a pull request

Thank you!

Did you get any ideas on how to use the application?

References

Carter, N. (2021). *Visual group theory* (Vol. 32). American Mathematical Soc.

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Magnus, W., Karrass, A., & Solitar, D. (2004). *Combinatorial group theory: Presentations of groups in terms of generators and relations*. Courier Corporation.

Kobourov, S. (n.d.). G.,“force-directed drawing algorithms,” handbook of graph drawing and visualization, chapter 12, 2013.