On Train Tracks and Splitting Sequences

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Preface

This report was written as part of FrontierLab, an exchange programme run by Osaka University that places international students in research laboratories for up to 12 months.

As part of the programme, I joined the research group of Eiko Kin. Under her supervision I conducted mathematical research for 11 months. I visited various conferences, such as Topology and Computers 2023, Magnitude 2023, Four Dimensional Topology 2023, The 19th East Asian Conference on Geometric Topology, the MSJ Spring Meeting, and the Intelligence of Low-dimensional Topology 2024. I attended the topology seminar at Osaka University which gave me many opportunities to read literature outside my own field.

I developed the application Graphicayley (to be found at `https://www.graphicayley.de/`). The application is used to draw Cayley graphs from a set of generators and relators. I presented the application at the East Asian Conference of Geometric Topology and then at the Illustrating Mathematics Seminar. The application is now regularly used by geometric group theorists in Japan, the USA and Germany.

In particular, together with Eiko Kin, I carried out research on train tracks and Agol cycles. During my research I read the available literature, presented important concepts in the research seminar, drew a lot of train track diagrams and made some minor changes to Issa's Veering software. At the What Is Seminar, I had the chance to explain the basics of train tracks and Agol cycles to a broad mathematical audience. The talk lasted 45 minutes long and was held in Japanese. The knowledge and my own research results will be the main focus of this report. Some results will not be discussed. Instead they will appear in a paper by Eiko Kin, to which I have contributed in an advisory capacity.

I want to thank Eiko Kin for her kind supervision and her continuous support during my stay in Japan. She encouraged me to present at the EACGT and at the What Is seminar. She constantly appreciated my work and always gave honest and helpful feedback. She helped me to set my priorities straight when I was working on too many things at once. Thanks to her, I have learned learned a lot and grown as a mathematician.

Introduction

Pseudo-Anosov maps are maps on a surface that appear chaotic, in the sense that a small disc on the surface is stretched in one direction and contracted in the other by a factor called dilatation, causing most nearby points to diverge rapidly as the map is iterated. Nevertheless, pseudo-Anosov maps contain a lot of interesting structure because they set-wise fix two transverse, measured geodesic laminations. [Thu88].

Train tracks were first introduced by Thurston (see [Thu79]). They are C^1-embeddings of graphs into a surface where the edges of a vertex all meet in a tangent line and also have some additional properties. We can assign weights to each edge, which gives us a measured train track. Up to shifting, folding and splitting operations, the measured train track corresponds to a measured geodesic lamination on the surface. Therefore each pseudo-Anosov map preserves an equivalence class of invariant train tracks which can be used to study the transverse measured foliation and the pseudo-Anosov map combinatorially [PH91].

For a given pseudo-Anosov map, Ian Agol studied the splitting sequences of these invariant train tracks. The splitting sequence being an object resulting from the repeated application of so-called maximal splittings at the edges with the largest weight. He noticed that these sequences eventually lead to a cycle. Each time a cycle is realised, the train track is changed by the pseudo-Anosov map and the weights are scaled by the inverse of the dilatation [Ago11].

It is a relatively well-known result of Mosher [Mos03b] that projective train tracks on the torus can be identified with continued fractions, with the maximal split acting as a kind of continued fraction transformation map $x \mapsto 1/(x - \lfloor x \rfloor)$. This was further been investigated and generalised to the 3-punctured disc by Aceves, Kin, Kawamuro [AK23, KK23]. In a pretty recent article on the AMS [Mar23], Dan Margalit describes a conjecture, first raised by Fried [Fri85], that all stretch factors are bi-Perron units, implying a deeper connection between number theory and low-dimensional topology. However, Margalit also mentioned that this connection is still not very clear. In light of this, we want to strengthen the connection. We give some evidence for the fact that the connection between train tracks of the torus and continued fractions can be generalized to other surfaces if Perrons's multidimensional continued fractions [Per07] are used.

The report is structured as follows. First, we give some background on the theory of continued fractions, train tracks and splitting sequences. We then do a quick literature review, to scout out some similar approaches and find good introductory material for the curious. Finally, we identify train tracks with continued fractions, first for a family of pseudo-Anosov maps on the once-punctured torus and then on the twice-punctured torus. We describe how the maximal split acts on the continued fraction and give a formula for the dilatation, introduced by [Bau92]. If time allows, we will try to find a generalisation for maps given by Penner's construction.

This report is based on joint work with Eiko Kin.

Chapter 1

Background

We give some background necessary for the theory explained later on. The document was intended to require only knowledge of topology but due to time constraints, some important details will be omitted and referred to the literature instead. To be precise, we won't go into the details of Perron-Frobenius theory, the reader is advised to read [Kit12] or one of the many other books in the subject instead. We won't go into laminations and pseudo-Anosov maps but good explanations can be found in [Thi22, Iss12, Mar18].

1.1 Continued fractions

We give a brief introduction to continued fractions, their properties and provide a pictorial characterisation in the form of a rectangle. (An explanation can be found in [Kno], an earlier description can be found in [Kim83].

The introduction to continued fractions will follow [HW08] as it is considered a classic in the field. I will also borrow some aspects from [Bau92].

Definition 1.1.1 (Continued fraction)**.** Let $w \in \mathbb{Q}_+$ be a positive, rational number. We say, w has the *(unitary) continued fraction expansion* $[a_1, ..., a_l]$ if

$$w = a_1 + \frac{1}{a_1 + \frac{1}{... + \frac{1}{a_l}}}$$

with $a \in \mathbb{N}_0$

For this definition of the continued fraction we allow 0 as a coefficient. This means that a number can have multiple continued fraction expansions, such as $[a, 0, b] = [a + b]$ but it will be more natural to generalise later on.

Definition 1.1.2 (Infinite continued fraction)**.** Let $w \in \mathbb{R}_+$ be a positive, real number. We say, w has the *(unitary) infinite continued fraction expansion* $[a_1, a_2, ...]$ if

$$w = \lim_{t\to\infty} [a_1, ..., a_t]$$

If the continued fraction expansion eventually repeats, we write

$$w = [a_1, ..., a_n, \overline{b_1, ..., b_m}] = [a_1, ..., a_n, b_1, ..., b_m, b_1, ..., b_m, ...]$$

Continued fractions can be characterized using matrices. We first define equality up to rescaling.

Definition 1.1.3 (Slope)**.** The *slope* of a vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}_+^n$ is $s := \begin{pmatrix} v_2/v_1 \\ \vdots \\ v_n/v_1 \end{pmatrix}$

Definition 1.1.4 (Projective equivalence)**.** For two vectors $v, w \in \mathbb{R}_+^n$, we define the *projective equality*

$$v \overset{p}{=} w \iff v = cw$$

for a $c \in \mathbb{R}_{>0}$. Also, for a sequence of vectors $v_i \in \mathbb{R}_+^n$, we write

$$\lim_{t\to\infty} v_i \overset{p}{=} v$$

if we can find a sequence of $c_i \in \mathbb{R}_+$ s.t. $\lim_{t\to\infty} c_i v_i = v$.

Definition 1.1.5 (Projective equivalence class)**.** Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}_{>0}^n$ be a vector with slope $s = \begin{pmatrix} v_2/v_1 \\ \vdots \\ v_n \; v_1 \end{pmatrix}$. Then the *equivalence class of the projective equality* is denoted with its slope, written in square brackets $[s]_p$.

$$[s]_p := \left[\begin{pmatrix} 1 \\ v_2/v_1 \\ \vdots \\ v_n/v_1 \end{pmatrix}\right]_{\overset{p}{=}} = \left[\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right]_{\overset{p}{=}}$$

The space of equivalence classes is equal to $\mathbb{R}_+^{n-1}$

Geometrically, the projective equivalence classes are positive rays, emanating from the origin. Two vectors are projectively equivalent if and only if they have the same slope. We can say that a vector v has the slope s by writing $v \in [s]_p$. In the next step, we take the previous characterisation as our definition of continued fractions.

Theorem 1.1.6 (Lemma 1 in [Bau92])**.** *Let $s = [a_1, a_2, ...]$. Then the following holds:*

$$\lim_{t\to\infty} D[a_1]...D[a_t] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \overset{p}{=} \begin{pmatrix} 1 \\ s \end{pmatrix} \in [s]_p$$

where $D[a] = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$

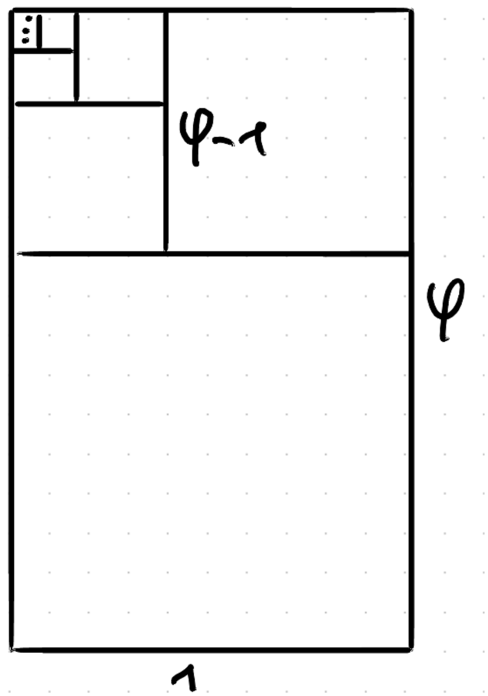


Figure 1.1: A golden ratio party trick

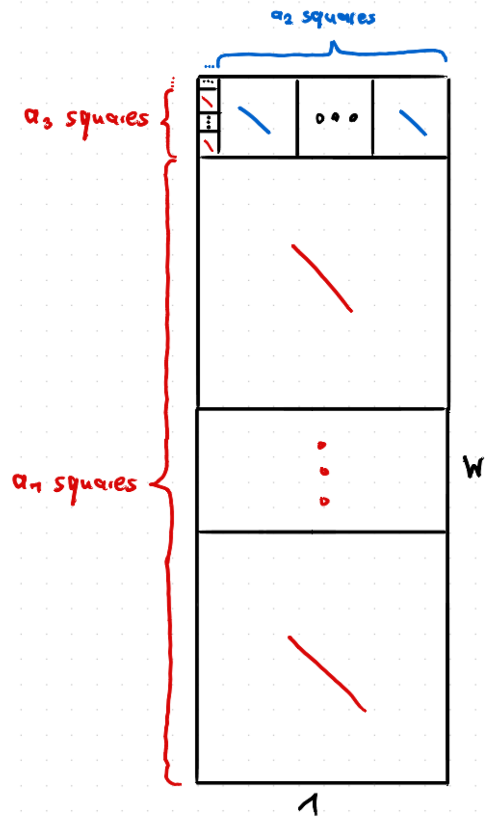


Figure 1.2: Continued fraction visualized as a rectangle

It is a well-known party trick that for a rectangle with the side ratio of the golden ratio $\varphi = [\overline{1}] = \frac{1+\sqrt{5}}{2}$, you can cut out a square and get a smaller rectangle with the side ratio being, once again, the golden ratio. This follows from the fact that $\frac{\varphi}{1} = \frac{1}{\varphi-1}$. The process of cutting off squares can be repeated indefinitely many times, giving us the figure 1.1.

This trick does not only apply to $\varphi = [\overline{1}]$, but also extends to other continued fractions $w = [a_1, a_2, ...]$. If we draw a rectangle with side ratio w, we can cut off at most a_1 squares from the left. From the resulting rectangle we can then cut off a_2 squares from the bottom. Then we can cut off a_3 squares from the left again, and so on. Furthermore, the sequence of how many square we cut off determines the side ratio of the rectangle.

Definition 1.1.7. $\mathrm{rect}\binom{v_1}{v_2}$ denotes a *rectangle* of width v_1 and height v_2. We say, a rectangle *admits a square partition* $[a_1, a_2, a_3, ...]$ if it can be partitioned into squares, as seen in figure 1.2.

Theorem 1.1.8 ([Kno])**.** *For $v = \binom{v_1}{v_2}$, a rectangle $\mathrm{rect}(v)$ admits a square partition $[a_1, a_2, a_3, ...]$ if and only if it v has slope $s = [a_1, a_2, a_3, ...]$.*

1.2 n-ary continued fractions

We review the definition of the n-ary continued fraction as described in [Bau92]. We first describe the normal continued fraction and then we introduce a characterisation taken from linear algebra.

Definition 1.2.1 (n-ary continued fraction)**.** We generalise the arguments of the continued fraction definition as follows:

$$\begin{aligned} a_i \in \mathbb{N}_0 &\implies y^{(i)} \in \mathbb{N}_0^n \\ D(a_i) \in M_2(\mathbb{N}_0) &\implies D_n(y^{(i)}) \in M_{n+1}(\mathbb{N}_0) \\ s = [a_1, ..., a_t] \in \mathbb{Q} &\implies w = [y^{(i)}, ..., y^{(t)}] \in \mathbb{Q}^n \end{aligned}$$

A vector $s = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{Q}_+^n$ has the *(finite) n-ary continued fraction expansion* $[y^{(1)}, ..., y^{(t)}]$ if

$$[s]_p = \left[D_n[y^{(1)}]...D_n[y^{(t)}] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right]_{\underline{p}}$$

where $D_n[y^{(i)}] = \begin{pmatrix} 0 & 1 \\ \mathbb{I}_n & y^i \end{pmatrix} = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & y_1^{(i)} \\ 0 & 1 & ... & 0 & y_2^{(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & ... & 1 & y_n^{(i)} \end{pmatrix}$

A vector $s = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \in \mathbb{R}_+^n$ has the *(infinite) n-ary continued fraction expansion* $[y^{(1)}, y^{(2)}, ...]$ if

$$[s]_p = \lim_{t \to \infty} \left[D_n[y^{(1)}]...D_n[y^{(t)}] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right]_{\underline{p}}$$

Once again, we write:

$$[y^{(1)}, ..., y^{(n)}, \overline{z^{(1)}, ..., z^{(m)}}] := [y^{(1)}, ..., y^{(n)}, z^{(1)}, ..., z^{(m)}, z^{(1)}, ..., z^{(m)}, ...]$$

Note the use of language here. Instead of defining the value of an n-ary continued fraction, we have described when a vector $w \in \mathbb{R}_+^n$ has an n-ary continued fraction expansion. This was done because, unlike to the unitary case, the infinite n-ary continued fraction $[y^{(1)}, ...]$ for vectors $y^{(i)} \in \mathbb{N}_0$ does not always converge to a well-defined value. In our case, this won't be an issue.

Next, we want to invert the above formulas to get a condition for when w has a continued fraction expansion. For the finite case this is straightforward but the infinite case requires some extra work.

Proposition 1.2.2. *$s \in \mathbb{Q}_+^n$ has the finite n-ary continued fraction expansion if and only if*

$$D_n[y^{(t)}]^{-1}...D_n[y^{(1)}]^{-1} \begin{pmatrix} 1 \\ s \end{pmatrix} \stackrel{p}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$s \in \mathbb{R}_+^n$ has the infinite n-ary continued fraction expansion if and only if

$$\lim_{t\to\infty} D_n[y^{(t)}]^{-1}...D_n[y^{(1)}]^{-1}\begin{pmatrix}1\\ s\end{pmatrix} = 0$$

Proof.

$$[s]_p = \left[D_n[y^{(1)}]...D_n[y^{(t)}]\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\right]_{\overset{p}{=}} \iff \begin{pmatrix}1\\ s\end{pmatrix} \overset{p}{=} D_n[y^{(1)}]...D_n[y^{(t)}]\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}$$

therefore the finite case results from inverting the matrices $D_n[y^{(i)}]$.
The infinite case is trickier. $[s]_p = \lim_{t\to\infty}\left[D_n[y^{(1)}]...D_n[y^{(t)}]\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix}\right]_{\overset{p}{=}}$ means, there is a sequence $(c_t)_{t\in\mathbb{N}}, c_t \in \mathbb{R}_+$ converging to zero such that

$$\begin{pmatrix}1\\ s\end{pmatrix} = \lim_{t\to\infty} c_t D_n[y^{(1)}]...D_n[y^{(t)}]\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} = \lim_{t\to\infty} D_n[y^{(1)}]...D_n[y^{(t)}]\begin{pmatrix}0\\ \vdots\\ 0\\ c_t\end{pmatrix}$$

From this we get

$$\lim_{t\to\infty} D_n[y^{(t)}]^{-1}...D_n[y^{(1)}]^{-1}\begin{pmatrix}1\\ s\end{pmatrix} = \lim_{t\to\infty}\begin{pmatrix}0\\ \vdots\\ 0\\ c_t\end{pmatrix} = 0$$

□

We continue by describing the eigenvalues of products of $D_n[y^i]$-matrices explicitly:

Theorem 1.2.3 (Theorem 6 in [Bau92])**.** *For $y^{(1)}, ..., y^{(p)} \in \mathbb{N}_0^n$ let $M = D_n[y^{(i)}]...D_n[y^{(p)}] \in M_{n+1}(\mathbb{Z})$ be a matrix as seen in the definition of the n-ary continued fraction. Suppose that M is Perron-Frobenius. Then the Perron-Frobenius eigenspace of M is:*

$$[s]_p = \begin{bmatrix}s_1\\ \vdots\\ s_n\end{bmatrix}_p$$

where $s = [\overline{y^{(1)}, ..., y^{(p)}}]$. Its eigenvalue can be explicitly described as:

$$\lambda = \prod_{v=1}^{p}(T^{v-1}[\overline{y^{(1)}, ..., y^{(p)}}])|_n$$

where T is a shift operator $T[y^{(1)}, y^{(2)}, y^{(3)}, ...] = [y^{(2)}, y^{(3)}, ...]$ on the continued fraction and $|_n$ is the projection to the n-th coefficient.

Observe that the shift operator T is not well defined as a function of $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ but only as an operator on the continued fraction expansion, since $s \in \mathbb{R}_+^n$ can have many different continued fraction expansions.

1.3 Train tracks

A train track is a special kind of graph, embedded into a surface. It was introduced by Thurston [Thu79] and named by him and his students. Its name implies a structure, similar to a real railway track, which can be smoothly traversed by a train. Mathematically this "train traversal" will be realised by a differentiable curve α running on the train track at a non-vero speed.

The definitions in this chapter generally follow Issa's master thesis [Iss12]. This report will only be interested in trivalent train tracks as they are easier to work with. Since a non-trivalent train track is, in a sense, equivalent to a trivalent train track, it makes no difference later on. I will also only consider surfaces S with Euler characteristic $\chi(S) < 0$, since some definitions have a habit of breaking on the flat torus.

Remark 1.3.1. For the rest of the paper, $S = S_{g,p}$ will denote a surface of genus $g \geq 0$ with $p \geq 0$ punctures of Euler characteristic $\chi(S) < 0$. All maps $S \to S$ will be orientation-preserving homeomorphisms. Furthermore, when we speak of a homeomorphism f, we will be ambiguous as to whether we mean the mapping class of f or the homeomorphism f. Since we are interested in topological properties up to isotopy, this doesn't cause any problems but caution is advised.

Definition 1.3.2 (Train track)**.** A *train track* on a surface S is a connected graph $\tau \subseteq S$, embedded in S. We call the vertices *switches* and the edges *branches*. The train track is trivalent, i.e. each switch has three branches attached to it. We also need the the following properties [Iss12, PH91]:

(i) **Branch shape**: The interior of each branch b is c^1, meaning, there is a continuously differentiable curve $\alpha : [0, 1] \to S$ parametrising b

(ii) **Switch shape**: If s is a switch then it looks line in figure 1.3, i.e. there is a tangent line $T_s(\tau)$ sitting in the tangent space $T_s(S)$ at v. Additionally, we require at least one edge at each end of the tangent line. Meaning, if $\alpha[0, 1]$ is a curve parametrising the switch s and two connected branches with $\alpha(\frac{1}{2}) = s$ then $\frac{d\alpha}{dt}(\frac{1}{2}) \in T_s(\tau)$

(iii) **Geometrical condition**: We require the train track to be "essential". This means, no connected component of $S - \tau$ is an embedded disc, annulus, monogon or bigon.

The set of all branches is denoted $E(\tau)$, the set of switches is denoted $V(\tau)$

A train track $\tau \subseteq S$ is "essential" if all differentiable embeddings $c : S^1 \to \tau$ of a closed curve into the train track are essential closed curves. [FM11] defines essential closed curves as follows:

Definition 1.3.3 (Essential closed curve)**.** Let S be a surface. An *essential closed curve* is a closed curve on S which is not homotopic to a point, a puncture or a boundary component.

A Dehn-Twist around a non-essential simple closed is homotopic to the identity and therefore not very interesting for the study of mapping class groups. This is why we will require our closed curves to be essential.

Definition 1.3.4 (Measured train track)**.** A *measured train track* is a train track on a surface S with a measure $\mu : E(\tau) \to \mathbb{R}_{>0}$, assigning positive weights to the edges of the train track, such that the *switch condition* (see figure 1.4) holds.

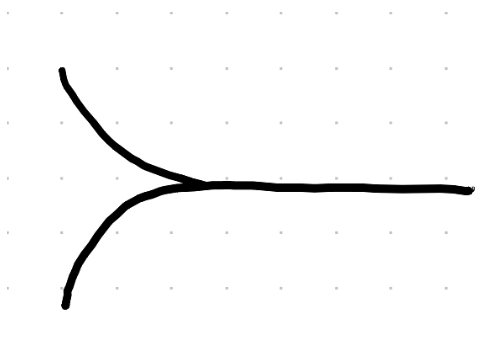


Figure 1.3: Train track switch

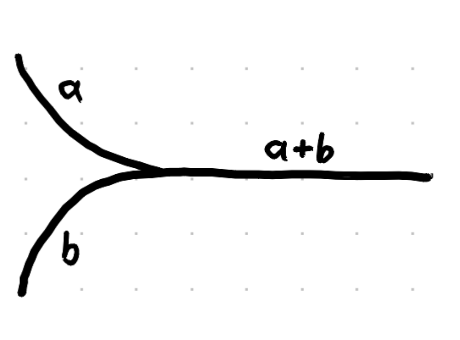


Figure 1.4: Switch condition

Often we will be interested in measured train tracks up to rescaling. We reuse the previous notation $[\cdot]_p$ to describe projective train tracks.

Definition 1.3.5 (Projective train track)**.** Let S be a surface. Two measures μ, μ' of a train tracks $\tau \subseteq S$ are considered to be *projectively equivalent* $\mu \overset{p}{=} \mu'$ if their measures differ by a positive factor, i.e. $\mu' = c\mu$ for a $c \in \mathbb{R}^+$.
The equivalence class of a measured train track (τ, μ) up to rescaling of the measure is called a *projective train track* and denoted $(\tau, [\mu]_p) := [(\tau, \mu)]_{\overset{p}{=}}$

Definition 1.3.6 (Pushforward)**.** Let (τ, μ) a measured train tracks and $f : S \to S$ be a homeomorphism. We define the *pushforward of* μ *with respect to* f as

$$f_*(\mu)(b) := \mu(f^{-1}(b))$$

for a branch $b \in E(f(\tau))$. Furthermore, we write

$$f(\tau, \mu)(b) := (f(\tau), \mu(f^{-1}(b)))$$

This descends to a pushforward for projective train tracks.

We define equality and combinatorial equivalence to projective train tracks. This is inspired by [Mos03b].

Definition 1.3.7 (Equality for train tracks)**.** Let $\tau, \tau' \subseteq S$ be two train tracks and μ, μ' measures on the two train tracks respectively.

- The train tracks τ, τ' are called *equal* (or *isotopic*) if $f(\tau) = \tau'$.
- The measured train tracks $(\tau, \mu), (\tau', \mu')$ are called *equal* (or *isotopic*) if $f(\tau, \mu) = (\tau', \mu')$.
- The projective train tracks $(\tau, [\mu]_p), (\tau', [\mu']_p)$ are called *equal* (or *isotopic*) if $f(\tau, \mu) \in (\tau', [\mu']_p)$

for a homeomorphism f, isotopic to the identity. The equality is denoted symbolically by $=$.

Definition 1.3.8 (Combinatorial equivalence for train tracks)**.** Let $\tau, \tau' \subseteq S$ be two train tracks and μ, μ' measures on the two train tracks respectively.

- The train tracks τ, τ' are called *combinatorially equivalent with respect to f* if $f(\tau) = \tau'$.
- The measured train tracks $(\tau, \mu), (\tau', \mu')$ are called *combinatorially equivalent with respect to f* if $f(\tau, \mu) = (\tau', \mu')$.
- The projective train tracks $(\tau, [\mu]_p), (\tau', [\mu']_p)$ are called *combinatorially equivalent with respect to f* if $f(\tau, \mu) \in (\tau', [\mu']_p)$

for an orientation-preserving homeomorphism f. The combinatorial equivalence is denoted symbolically by $\cong$.

The next definition is from [HIS16].

Definition 1.3.9 (Suited train track)**.** Let S be a surface. Let $(\mathcal{L}, \lambda)$ be a measured geodesic lamination on S and let (τ, μ) be a measured train track on S. The measured train track (τ, μ) is suited to $(\mathcal{L}, \lambda)$ if the following conditions are satisfied

- There exists a differentiable, non-homeomorphism $\phi : S \to S$ homotopic to the identity such that $f(\mathcal{L}) = \tau$
- If $\alpha : [0, 1] \to \tau'$ is a curve with everywhere non-zero velocity parametrizing a finite subset of a leaf of $\mathcal{L}$ then $\phi \circ \alpha : [0, 1] \to \tau$ too has everywhere non-zero velocity.
- ϕ respects the transverse measures, that is if p is a point in the interior of an edge $b \in E(\tau)$ then $\lambda(\phi^{-1}(p)) = \mu(e)$

Definition 1.3.10 (Carrying train tracks)**.** Let $\tau, \tau' \subseteq S$ be train tracks. τ *carries* τ' if there exists a *carrying map* collapsing τ' onto τ, that is, a C^1-map ϕ homotopic to the identity, satisfying:

(i) $\phi(\tau') \subset \tau$

(ii) If $s \in V(\tau')$ is a switch of τ', then $\phi(s)$ is a switch of τ

(iii) If $\alpha : [0,1] \to \tau'$ is a curve with everywhere non-zero velocity then $\phi \circ \alpha : [0,1] \to \tau$ too has everywhere non-zero velocity.

Definition 1.3.11 (Invariant train tracks)**.** Let $f : S \to S$ be a homeomorphism of a surface S. A train track $\tau \subseteq S$ is *invariant with respect to* f if τ carries $f(\tau)$.

Let τ be invariant with respect to f. Looking at one branch $b \in V(\tau)$ and applying f and then the carrying map ϕ, the two switches of b are mapped to switches of τ. Similarly, the branch b is mapped to a branch path (i.e. a sequence of branches). This leads us to define the *incidence matrix*, a matrix which describes how edges are mapped back into τ.

Definition 1.3.12 (Incidence matrix)**.** Let $\tau \subset S$ be a train track invariant with respect to a homeomorphism $f : S \to S$ and a carrying map ϕ. For τ, we choose a numbering and orientation for the edges $e_1, ..., e_n : [0,1] \to \tau$.
The *incidence matrix* (sometimes *transition matrix*) is the matrix $M = M(\tau, f)$ where $M_{i,j}$ is the number of times the curve $\phi \circ f(e_j)$ crosses e_i as it is traversed, that is,

$$M_{i,j} = |e_j \cup (\phi \circ f)^{-1}(e_i)| \quad 1 \leq 1, j \leq k$$

where $|A|$ indicates the number of connected components of A (edges do not include their endpoints)

The following theorem demonstrates the usefulness of the incidence matrix. If we have a train track that is invariant under f and an incidence matrix that satisfies certain conditions, then it is sufficient to conclude that f is pseudo-Anosov.

Theorem 1.3.13 (Theorem 3.4 of [Los93])**.** *Let $f : S \to S$ be a homeomorphisms of a surface S such that there exists an invariant train track $\tau \subseteq S$ satisfying the following conditions:*

(i) The train track τ is connected and fills the surface S, that is, each connected component of $S - \tau$ is topologicall homeomorphic to a disc or an annulus.

(ii) The indidence matrix M with respect to the edges $e_1, ..., e_k$ of τ has a real eigenvalue $\lambda > 1$ with and maximal modulus among all eigenvalues.

(iii) The eigenvalue has multiplicity 1 and its eigenspace is spanned by an eigenvector with strictly positive entries

(iv) Assigning the weights $v_1, ..., v_k$ to the branches $b_1, ..., b_k$ respectively, turns τ into an invariant measured train track.

then f is isotopic to a pseudo-Anosov homeomorphism with dilatation λ.

The proof of the theorem relies heavily on the Bestvina-Handel algorithm [BH95]. This algorithm takes a homeomorphism f as input and computes a train track that is invariant under f. Along the way, the algorithm determines whether f is pseudo-Anosov or not.

Theorem 1.3.14. *If a train track τ is invariant under g and f with incidence matrices M_g, M_f, it is invariant under $g \circ f$ as well. The incidence matrix is given by the product of the incidence matrices $M_{gf} = M_g M_f$.*

Proof. Among experts this theorem is relatively well known. Instead of giving a detailed proof, we will quickly make use of fibered neighbourhood. (See [Iss12] for more details) We will consider a small neighbourhood $N(\tau)$ of the train track τ. We will make the neighbourhood a fibered surface as shown in figure 1.5, so that the fibers are transverse to the train track.

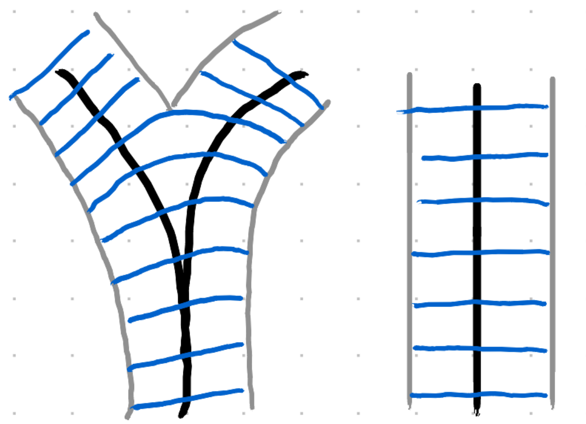


Figure 1.5: Fibered neighbourhood of a train track at switches and branches

The train track τ is invariant under f. W.l.o.g. we deform f so that $f(\tau) \subseteq N(\tau)$. We can also deform f such that $f(N(\tau)) \subseteq N(\tau)$, while also mapping fibers onto fibers. The same can be done for g.
This makes gf becomes fiber-preserving and the map satisfies $gf(N(\tau)) \subseteq N(\tau)$. By contracting the individual fibers to a point, we can easily construct a carrying map for gf. Since the train tracks remain transverse to the fibers under gf, the carrying map is well-defined and τ becomes invariant under gf. In particular, there are no "back-tracks" in the carrying map.

Let $E(\tau) = \{b_1, ..., b_n\}$. Describing a multi-set of branches using a vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $M_f x$ describes to which branches x is sent to under f. $M_g M_f x$ then describes to which branches x is sent to under $g \circ f$, proving the last claim. □

1.4 Penner's construction

In his 1988 paper Penner described a way to construct an entire semigroup of pseudo-Anosov map on a (possibly non-orientable) surface [Pen88]. We will quickly outline the construction for oriented surfaces here.

Definition 1.4.1 (Curves hitting efficiently)**.** Let c, d be two closed curves on an orientable surface S. We say, c *hits* d *efficiently* if $S - (c \cup d)$ has no bigon component.

Definition 1.4.2 (Penner's construction)**.** Let S be an oriented surface with negative Euler characteristic $\chi(S) \leq 0$. Let $\mathcal{C} = \{c_1, ..., c_n\}$ be a finite collection of disjoint, essential simple closed curves on S. Let $\mathcal{D} = \{d_1, ..., d_m\}$ be a collection of curves with the same properties. Suppose, $\mathcal{C} \cup \mathcal{D}$ fill S, are not pairwise parallel and the curves in $\mathcal{C}$ and $\mathcal{D}$ hit each other efficiently.
Let f_{L_i} denote the right-handed Dehn-twist around the curve c_i and f_{R_j} the left-handed Dehn-twist around the curve d_j. The semi-group generated by the Dehn-Twists is called *Penner's construction*:

$$F(f_{R_1}, ..., f_{R_n}, f_{L_1}, ..., f_{L_m}) := \langle f_{R_1}, ..., f_{R_n}, f_{L_1}, ..., f_{L_m} \rangle$$

Furthermore, we define the train track $\tau_{\mathcal{C},\mathcal{D}}$ by smoothing the edges as shown in figure 1.6. The collection of branches obtained from c_i (resp. d_j) by smoothing is denoted C_i (resp. D_j).

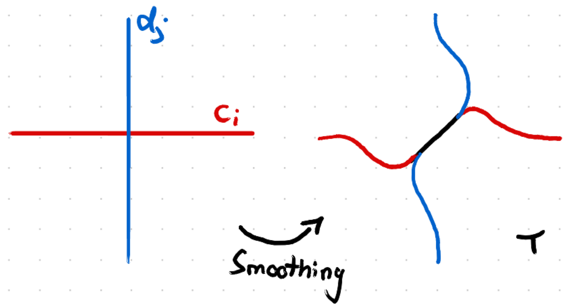


Figure 1.6: Smoothing crossings between curves c_i and d_j

We mention a side effect of the above definition. The large branch, obtained by smoothing the crossing between c_i and d_j, will be contained in both C_i and D_j. We will be able to adjust for this later. For the sake of simplicity, we will refer to C_i and D_j as curves.

We continue by analysing how Dehn-twists act on C_i and D_j.

Lemma 1.4.3. *Let $\mathcal{C} = \{c_1, ..., c_n\}, \mathcal{D} = \{d_1, ..., d_m\}$ be collection of curves as in Penner's construction. Then the train track $\tau = \tau_{\mathcal{C},\mathcal{D}}$ is invariant under the Dehn-twists f_{R_i}, f_{L_j}. Ordering the curves as follows: $(C_1, ..., C_n, D_1, ..., D_m)$ the map f_{L_i} and f_{R_j} respectively induce the incidence matrix:*

$$L_i = \begin{pmatrix} I_n & A \\ 0 & I_m \end{pmatrix} \text{ and } R_j = \begin{pmatrix} I_n & 0 \\ B & I_m \end{pmatrix}$$

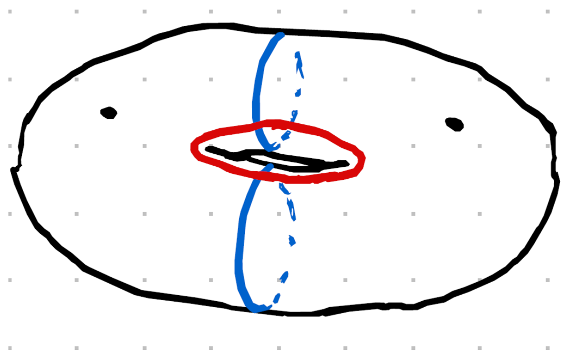


Figure 1.7: Example curves for Penner's construction

where

$$A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \#(c_i \cap d_{j_1}) & \dots & \#(c_i \cap d_{j_m}) \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & \dots & 0 & \#(c_{i_1} \cap d_j) & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & \#(c_{i_n} \cap d_j) & 0 & \dots & 0 \end{pmatrix}$$

With $\#(c \cap d)$ giving the count of intersections between the curves c and d

Proof. We study the effect of a right-handed Dehn-twist. f_{R_i} only affects the neighbourhood $S(c_i)$ of the curve c_i and is the identity outside.
After smoothing the curve $S(c_i)$ locally looks like the left part of figure 1.8, with f_{R_i} changing it into the right part. We can easily construct a carrying map to show that τ is invariant. The matrices follow from the figure.

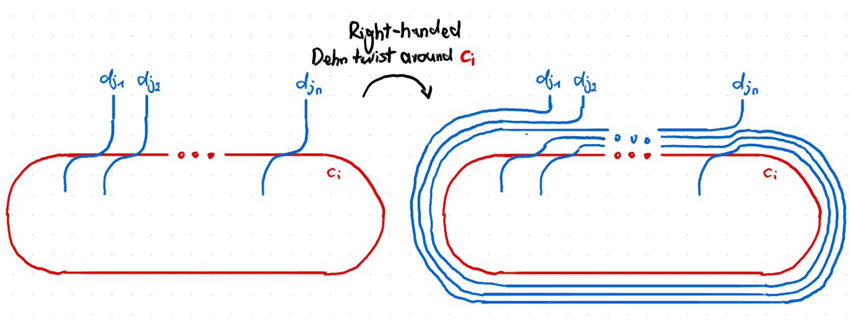


Figure 1.8: Right-handed Dehn-twist around c_i

The proof for f_{L_j} is analogous. □

There is a very important remark to be made. Since C_i, D_j are not branches but a collection of branches, the above is not really an incidence matrix in the sense defined previously. Nevertheless, the resulting matrix has very similar properties to the incidence matrix. In particular, f is pseudo-Anosov when the incidence matrix is Perron-Frobenius. We won't go into the details here. A similar treatment can be found in [Pen91].

Lemma 1.4.4. *Let $\mathcal{C} = \{c_1, ..., c_n\}, \mathcal{D} = \{d_1, ..., d_m\}$ be collection of curves as in Penner's construction. Then a map $f_{L_{i_n}}^{-1} f_{R_{i_n}} ... f_{L_{i_1}}^{-1} f_{R_{i_1}} \in F(f_{R_1}, ..., f_{R_n}, f_{L_1}, ..., f_{L_m})$ has the incidence matrix:*

$$M = L_{i_n} R_{j_n} ... L_{i_1} R_{j_1}$$

Proof. It is known from Lemma 1.4.3 that $\tau_{\mathcal{C},\mathcal{D}}$ is invariant under the Dehn-twists f_{L_i}, f_{R_j} with the incidence matrices being L_i, R_j. By Theorem 1.3.14, τ also becomes invariant under f. Its incidence matrix is $M = L_{i_n} R_{j_n} ... L_{i_1} R_{j_1}$.

□

Theorem 1.4.5 (Theorem 3.1 in [Pen88])**.** *Let $\mathcal{C} = \{c_1, ..., c_n\}, \mathcal{D} = \{d_1, ..., d_m\}$ be collection of curves as in Penner's construction. Then the map*

$$f_{L_{i_n}}^{-1} f_{R_{i_n}} ... f_{L_{i_1}}^{-1} f_{R_{i_1}} \in F(f_{R_1}, ..., f_{R_n}, f_{L_1}, ..., f_{L_m})$$

written as a word in the generators, is pseudo-Anosov if every generator appears at least once.

We will investigate example constructions in section 3

1.5 Agol cycles

We give a background on Agol cycles. This section will follow work by Ahmad Issa [Iss12].

Definition 1.5.1 (Small and large branches)**.** Let τ be a train track on a surface. A branch $b \in E(\tau)$ is called *small* if it looks like the figure 1.9 and *large* if it looks like the figure 1.10. Let $\mu : E(\tau) ø \mathbb{R}_+$ be a measure on the branches of τ. A branch $b \in E(\tau)$ is called *maximal* if it is large and has the largest weight of all branches.

Definition 1.5.2 (Split)**.** Let (τ, μ) be a measured train track on a surface S. Let $b \in E(\tau)$ be a large branch. A *split* separates the branch into two branches and, if necessary, connects them with a equalising branch, thus preserving the switch condition.
This produces a new measured train track (τ', μ'). The split of the train track at the branch (τ, μ) is denoted as:

$$(\tau, \mu) \rightharpoonup_b (\tau', \mu')$$

If the equalising branch is left-facing, we call the split a left split and write $\overset{L}{\rightharpoonup}_b$. If the branch is right-facing, we call the split a right split and write $\overset{R}{\rightharpoonup}_b$. Theoretically, there may be cases where there is no need to insert an equalising branch. But this case won't be relevant to our report, so we'll ignore it.

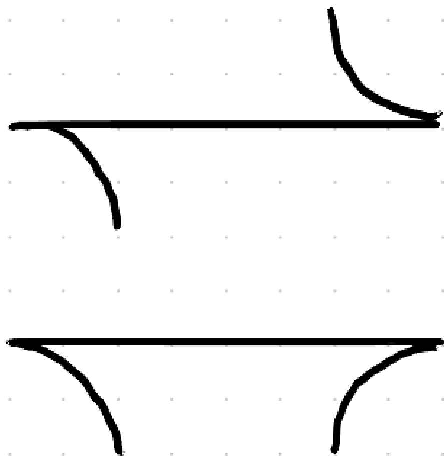


Figure 1.9: Small Branches

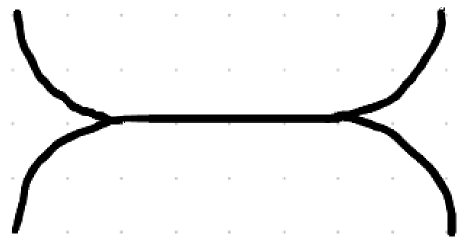


Figure 1.10: Large Branch

For two measured train tracks $(\tau, \mu), (\tau', \mu')$, we say that (τ, μ) splits into (τ', μ') if there is a sequence of branches and splits connecting the two train tracks.

$$(\tau, \mu) \rightharpoonup_{b_1} \dots \rightharpoonup_{b_n} (\tau', \mu')$$

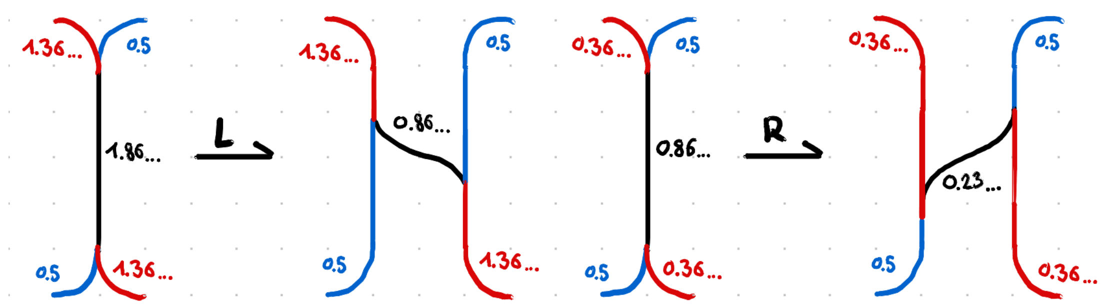


Figure 1.11: Left and right split of a branch with example weights

Notice, that the left split $\overset{L}{\rightharpoonup}_b$ and the right split $\overset{R}{\rightharpoonup}_b$ preserve the number of branches whereas $\overset{N}{\rightharpoonup}_b$ reduces the number. Since we want to keep the number of branches the same, $\overset{N}{\rightharpoonup}_b$ will not be important in this paper and we will make sure that it never occurs.

Definition 1.5.3 (Maximal Split)**.** Let (τ, μ) be a measured train track on a surface S. A *maximal split* splits all branches with the highest weights, producing a new measured train track (τ, μ). A maximal split is denoted as

$$(\tau, \mu) \rightharpoonup (\tau', \mu')$$

Sometimes we are interested in whether the splits are left or right splits. If all splits are in a maximal split are left splits, we write $\overset{L}{\rightharpoonup}$. If all splits are right, then we write $\overset{R}{\rightharpoonup}$. If both occur, we write $\overset{LR}{\rightharpoonup}$. In some cases, it is more convenient to write the maximal split as an operator on the measured train track $\rightharpoonup (\tau, \mu) = (\tau', \mu')$. Note that during concatenation, the order of the split symbols will be reversed!
A sequence of maximal splits

$$(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots \rightharpoonup (\tau_n, \mu_n)$$

will be shortened as $(\tau_0, \mu_0) \rightharpoonup^n (\tau_n, \mu_n)$. We also write $(\tau_0, \mu_0) \rightharpoonup^* (\tau_n, \mu_n)$ if the n is not of importance.

Definition 1.5.4 (Splitting sequence)**.** Let (τ_0, μ_0) be a measured train track. The infinite sequence of maximal splits

$$(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots$$

is called the *splitting sequence* of (τ_0, μ_0).

We define the periodic splitting sequence. This is a splitting sequence that is periodic modulo the action of a homeomorphisms and the rescaling of the measured train track by a factor. Note, that this definition is more general than the definition given in [HIS16]. There, a splitting sequence is called periodic if and only if it results from the theorem 1.5.12.

Definition 1.5.5 (Periodic splitting sequence)**.** Let (τ_0, μ_0) be a measured train track and

$$(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots$$

its splitting sequence. If there is a $n \in \mathbb{N}$ such that $(\tau_i, [\mu_i]_p)$ and $(\tau_{i+n}, [\mu_{i+n}]_p)$ are combinatorically equivalent with respect to a homeomorphism $f : S \to S$ for all $i \in \mathbb{N}$, then the splitting sequence is called *periodic* with respect to f with *period* n. If n is also minimal, it is called the *minimal period* of the splitting sequence.

Lemma 1.5.6 ([KK23])**.** *Let (τ, μ) be a measured train track in S. Let $\phi : S \to S$ be en orientation-preserving homeomorphisms. Then the split and homeomorphism commute.*

$$(\phi \circ \overset{l}{\rightharpoonup})(\tau, \mu) = (\overset{l}{\rightharpoonup} \circ \phi)(\tau, \mu)$$

The same statement holds for $\overset{r}{\rightharpoonup}$ and $\overset{lr}{\rightharpoonup}$

Lemma 1.5.7. *Let (τ_0, μ_0) be a measured train track and $(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots$ its splittings sequence. If there is a $n \in \mathbb{N}$ such that $(\tau_0, [\mu_0]_p) \cong (\tau_n, [\mu_n]_p)$ with respect to a homeomorphism $f : S \to S$, then the splitting sequence is periodic with respect to f and period n.*

Proof. We will show $(\tau_i, [\mu_i]_p) \cong (\tau_{i+n}, [\mu_{i+n}]_p)$ for all $i \in \mathbb{N}$ by induction. The base case is already given. We show $(\tau_{i+1}, [\mu_{i+1}]_p) \cong (\tau_{i+1+n}, [\mu_{i+1+n}]_p)$:

$$(\tau_{i+1+n}, [\mu_{i+1+n}]_p) = \rightharpoonup (\tau_{i+n}, [\mu_{i+n}]_p) = (\rightharpoonup \circ f)(\tau_i, [\mu_i]_p) = (f \circ \rightharpoonup)(\tau_i, [\mu_i]_p) = f(\tau_{i+1}, [\mu_{i+1}]_p)$$

by lemma 1.5.6, proving that the splitting sequence of (τ_0, μ_0) is periodic with respect to f. □

Lemma 1.5.8. *Let (τ_0, μ_0) be a measured train track and $\mathcal{S}$ its splittings sequence $(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots.$ Assume $\mathcal{S}$ is periodic with respect to a homeomorphism f with period.*
If there are two measured train tracks $(\tau_{m_1}, \mu_{m_1}), (\tau_{m_2}, \mu_{m_2})$ (with $m_2 > m_1$) such that $(\tau_{m_1}, [\mu_{m_1}]_p) \cong (\tau_{m_2}, [\mu_{m_2}]_p)$ with respect to a homeomorphism g, where $gf = fg$ then $\mathcal{S}$ is periodic with respect to g and period $m := m_2 - m_1$.

Proof. W.l.o.g. we assume $n > m_2$, otherwise we choose sufficient a power of f as period of $\mathcal{S}$. The subsequence of $\mathcal{S}$, beginning at (τ_{m_1}, μ_{m_1}) is periodic with respect to g, thereby we have $\rightharpoonup^m (\tau_n, [\mu_n]_p) = g(\tau_n, [\mu_n]_p)$. From this, we then get:

$$\begin{aligned} \rightharpoonup^m (\tau_0, [\mu_0]_p) &= (\rightharpoonup^{m+n} \circ f^{-1})(\tau_0, [\mu_0]_p) = (f^{-1} \circ \rightharpoonup^{m+n})(\tau_0, [\mu_0]_p) \\ &= (f^{-1} \circ \rightharpoonup^m)(\tau_n, [\mu_n]_p) = (f^{-1} \circ g)(\tau_n, [\mu_n]_p) \\ &= (f^{-1} \circ g \circ f)(\tau_0, [\mu_o]_p) = g(\tau_0, [\mu_o]_p) \end{aligned}$$

□

The assumption $fg = gf$ seems out of place but it is actually very natural for splitting sequences. If the splitting sequence of (τ_0, μ_0) is periodic with respect to f with period n and periodic with respect to g with period m then we have $(g \circ f)(\tau_0, [\mu_0]_p) = \rightharpoonup^{m+n} (\tau_0, [\mu_0]_p) = (f \circ g)(\tau_0, [\mu_0]_p)$, proving that the action of the two maps is commutative. If the train track (τ_0, μ_0) fills the surface, then the commutative action on the train track leads to a commutativity of the mapping classes. This means that if our train track has nice properties then the commutativity does not need to be a requirement for the above lemma.

Lemma 1.5.9. *Let (τ_0, μ_0) be a measured train track and $\mathcal{S}$ its splittings sequence. Let $f_1, f_2 : S \to S$ be homeomorphisms with $f_1 f_2 = f_2 f_1$. If $\mathcal{S}$ is periodic with respect to f_1 with period $n_1 \in \mathbb{N}$ and periodic with respect to f_2 with period n_2 then $\mathcal{S}$ is periodic with respect to a homeomorphism g with period $m = gcd(n_1, n_2)$ such that $g^{n_1/m} = f_1$ and $g^{n_2/m} = f_2$.*

Proof. We give some background. Euclid's algorithm can compute the greatest common divisor of two numbers n_1, n_2 in a finite amount of steps by repeatedly subtracting the smaller number from the larger. We will use it here to find m and g.
W.l.o.g. we assume $n_1 < n_2$. By lemma 1.5.8, the equation

$$\rightharpoonup^{n_2 - n_1} (\tau_{n_1}, [\mu_{n-1}]_p) = (\rightharpoonup^{n_2 - n_1} \circ \rightharpoonup^{n-1} \circ f_1^{-1})(\tau_{n_1}, [\mu_{n-1}]_p) = (g \circ f_1^{-1})(\tau_{n_1}, [\mu_{n-1}]_p)$$

shows that $\mathcal{S}$ is periodic with respect to $f_2 f_1^{-1}$ with period $n_2 - n_1$. This new map is commutative with both f_1 and f_2 as

$$f_2 f_1^{-1} \circ f_1 = f_2 f_1 f_1^{-1} = f_1 \circ f_2 f_1^{-1} \text{ and } f_2 f_1^{-1} \circ f_2 = f_2 f_2 f_1^{-1} = f_2 \circ f_2 f_1^{-1}$$

By induction on the step count, it can be shown that Euclid's algorithm returns a $m = gcd(n_1, n_2)$ and a homeomorphism g such that $\mathcal{S}$ is periodic with respect to g and with period m. Furthermore, $g^{n_1/m} = f_1$ and $g^{n_2/m} = f_2$. □

Agol showed that the splitting sequences of a two measured train tracks suited to the same stable lamination of a pseudo-Anosov map f have a peculiar property. After enough maximal splits they will both become the same measured train track (up to rescaling of the measure and isotopy)

Theorem 1.5.10 ([Ago11], Corollary 3.4)**.** *Let $(\mathcal{L}^s, \mu_s)$ be a measured lamination suited to the measured train tracks $(\tau_1, \mu_1), (\tau_2, \mu_2)$. Then there is a train track (τ, μ) such that $(\tau_1, \mu_1) \rightharpoonup^* (\tau, \mu)$ and $(\tau_2, \mu_2) \rightharpoonup^* (\tau, \mu)$.*

This implies that the splitting sequence of equivalent measured train tracks will eventually have the same tail. We formalise the idea of having "the same tail" by *combinatoric isomorphism.* If the measured train track is suited to the lamination of a map f, then its tail will even be periodic.

Definition 1.5.11. Let S be a surface. Let $(\tau, \mu), (\tau, \mu)$ be two measured train tracks with splitting sequences

$$(\tau_0, \mu_0) \rightharpoonup (\tau_1, \mu_1) \rightharpoonup \ldots \text{ and } (\tau_0', \mu_0') \rightharpoonup (\tau_1', \mu_1') \rightharpoonup \ldots$$

respectively. We denote the splitting sequences by $\mathcal{S}, \mathcal{S}'$ respectively. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *combinatorically isomorphic* if there are $t, s \in \mathbb{N}_0$ such that:

$$(\tau_t, [\mu_t]_p) \cong (\tau_s', [\mu_s']_p)$$

for all $i \in \mathbb{N}_0$

The above definition has been taken from [HIS16] but has a few important distinctions, but has been adapted to non-periodic splitting sequences.

Theorem 1.5.12 ([Ago11])**.** *Let $f : S \to S$ be a pseudo-Anosov map with dilatation λ and unstable measured lamination $(\mathcal{F}^u, \mu_u)$. If $(\mathcal{F}^u, \mu_u)$ is suited to the measured train track (τ, μ) then the splitting sequence of (τ, μ) is eventually periodic. More precisely, there exist $n, m \in \mathbb{N}_0$ such that*

$$(\tau, \mu) \rightharpoonup^n (\tau_n, \mu_n) \rightharpoonup^m f(\tau_n, \lambda^{-1}\mu_n)$$

Definition 1.5.13. Let $f : S \to S$ be a pseudo-Anosov map with dilatation λ and unstable measured lamination $(\mathcal{F}^u, \mu_u)$, suited to a measured train track (τ_0, μ_0).
Assume, the splitting sequence of (τ_0, μ_0) is periodic with $n \in \mathbb{N}$ such that $(\tau_0, \mu_0) \rightharpoonup^n f(\tau_0, \lambda^{-1}\mu_0)$. The finite subsequence

$$(\tau_0, \mu_0) \rightharpoonup \ldots \rightharpoonup f(\tau_0, \lambda^{-1}\mu_0)$$

is called an *Agol cycle with respect to* f.

- $l(f) := m$ is called the *length of the Agol cycle*.
- $N(f) := \sum_{i=1}^{m} \#(\text{Number of maximal branches in } (\tau_{n+i}, \mu_{n+i}))$ is called the *total splitting number*.

A map f usually has many related Agol cycles, so it is not clear whether the length or the total splitting number is well-defined or not. Furtunately, [Ago11] showed that any two measured train tracks which are suited to the stable lamination of f have combinatorically isomorphic splitting sequences. Their periodic parts differ only by a homeomorphism and a scaling factor, i.e. the Agol cycle length and the total splitting number are equal.

We also note the following. Let f and hfh^{-1} be a homeomorphism and a conjugate respectively. If (τ, μ) is a measured train track suited to the stable lamination of f, then $h(\tau, \mu)$ is a measured train track suited to the stable lamination of hfh^{-1}. From this follows that f and hfh^{-1} have combinatorically isomorphic splitting sequences and their Agol cycles will differ by a homeomorphism, a factor and a shift.

Definition 1.5.14 ([HIS16], Definition 5.1)**.** Let S be a surface. Let $f, f' : S \to S$ be pseudo-Anosov homeomorphisms with Agol cycles

$$(\tau_0, \mu_0) \rightharpoonup^m (\tau_m, \mu_m) = f(\tau_0, \lambda^{-1}\mu_0)$$
$$(\tau_0, \mu_0) \rightharpoonup^m (\tau_m, \mu_m) = f'(\tau_0, \lambda'^{-1}\mu_0)$$

respectively, where $n, m \in \mathbb{N}$ and $\lambda, \lambda' \in \mathbb{R}_{>0}$ is the dilatation of f, f' respectively. We denote the Agol cycles by $\mathcal{S}, \mathcal{S}'$ respectively. We say that $\mathcal{S}$ and $\mathcal{S}'$ are *combinatorically isomorphic* if:

1. They have the same length: $n = m$
2. The Agol cycles differ by a shift s, a homeomorphism h and a scaling factor $c \in \mathbb{R}_{>0}$
$$h(\tau_{i+s}, c\mu_{i+s}) = (\tau'_i, \mu'_i) \text{ or } h(\tau_i, c\mu_i) = (\tau'_{i+s}, \mu'_{i+s})$$
for all $i \in \mathbb{N}_0$

Proposition 1.5.15 ([HIS16], Theorem 5.3; [KK23], Theorem 1.4)**.** *Let $f : S \to S$ be a pseudo-Anosov map with dilatation λ and unstable measured lamination $(\mathcal{F}^u, \mu_u)$, suited to a measured train track (τ_0, μ_0). Then the splitting sequence of (τ_0, μ_0) up to combinatoric isomorphism is a conjugacy invariant of f.*

Suppose $(\tau_0, \mu_0) \rightharpoonup^n (\tau_n, \mu_n)$ is an Agol cycle with respect to f. Then the Agol cycle up to combinatoric isomorphism is a conjugacy invariant of f.

Proof. The previous remark already explained how the splitting sequences of two conjugates f, g are combinatorically isomorphic, since they eventually differ only by a homeomorphism and a scaling factor. They are eventually periodic with respect to f, g and with period n. The Agol cycles are the finite subsequences constituting the period of the spitting sequences. It is clear that they differ by a shift, a homeomorphism and a scaling factor. □

The next theorem will be useful in detecting Agol cycles. The above definition requires that after an Agol cycle the weights change by the dilatation but this is actually not necessary. Just showing that the train track changes by f already implies that the weights change by the dilatation λ, thus forming an Agol cycle.

Theorem 1.5.16. *Let $f : S \to S$ be a pseudo-Anosov map with dilatation λ and unstable measured lamination $(\mathcal{F}^u, \mu_u)$, suited to a measured train track (τ, μ).*
Assume, the splitting sequence of (τ, μ) is periodic with respect to f and period m. If

$$(\tau_0, [\mu_0]_p) \rightharpoonup^n f(\tau_0, [\mu_0]_p) \text{ then } (\tau_0, \mu_0) \rightharpoonup ... \rightharpoonup (\tau_n, \mu_n) = f(\tau_0, \lambda^{-1}\mu_0)$$

i.e. the maximal splits form an Agol cycle with respect to f.

Proof. If $(\tau_0, \mu_0) \rightharpoonup ... \rightharpoonup (\tau_n, \mu_n) = f(\tau_0, c^{-1}\mu_0)$ for a $c \in \mathbb{R}_+$ is not an Agol cycle, then there exists a $m \in \mathbb{N}$, distinct from n, such that $(\tau_0, \mu_0) \rightharpoonup ... \rightharpoonup (\tau_m, \mu_m) = f(\tau_0, \lambda^{-1}\mu_0)$ is an Agol cycle. W.l.o.g. $n < m$. We then have

$$(f \circ \rightharpoonup^{n-m})(\tau_0, [\mu_0]_p) = (\rightharpoonup^{n-m} \circ f)(\tau_0, [\mu_0]_p) = \rightharpoonup^{n-m} (\tau_m, [\mu_m]_p) = (\tau_n, [\mu_n]_p) = f(\tau_0, [\mu_0]_p)$$

meaning, by 1.5.8, that $\mathcal{S}$ is periodic with respect to the identity id and with period $n - m$. By lemma 1.5.9, $\mathcal{S}$ also has period $gcd(m, n-m) = gcd(m, n)$ and there is a map g such that $g^{n/gcd(m,n)} = f$ and $g^{m/gcd(m,n)} = id$. From the last equality follows that g is a periodic map (in the sense of the Thurston-Nielsen classification). Therefore f also must be periodic, which contradicts the assumption that f is pseudo-Anosov. □

Chapter 2

Literature Review

2.1 n-ary continued fractions

The heart of our study of n-ary continued fractions will be the 1992 paper by Bauer [Bau92]. Bauer defines n-ary continued fractions, shows how they can be used to describe the eigenvectors of certain Perron-Frobenius matrices and illustrates their connection to train tracks.

The n-ary continued fractions themselves were first introduced first by Jacobi [JH68] 1868 for the case $n = 2$ and then further generalised by Perron [Per07] in 1907.

In particular, Perron's the article has been very influential with 276 citations on Semantic Scholar. Going through some of them, we find:

- Nikolaev, 2003 [Nik03]: This paper defines geodesics in hyperbolic space whose slope is given by a multidimensional continued fraction

- Nikolaev, 2013 [Nik13]: A connection between operator algebras of pseudo-Anosov maps and n-ary continued fractions is investigated.

- Battagliola, 2022 [BMS23]: This paper describes a way to visualize multidimensional continued fractions

According to Bauer the first systematic approach to studying continued fractions by integer matrices was been done by Hummel in 1940 [Hum40] but according to Semantic Scholar this paper wasn't very influential. Bauer's paper itself also had only 2 citations, as well.

2.2 Train tracks

Train tracks have originally been introduced by Thurston [Thu79] to study the structure of geodesic laminations. A relatively recent explanation of train tracks can be found in Mosher's PhD thesis. [Mos83] The first comprehensive exposition on train tracks was written by Penner and Harer [PH91] which gives a detailed definition, describes splits, folds and shifts and again describes the connection to measured geodesic laminations. With 508 citations, this book has become highly influential and an important reference for much train track-related work. Some important examples (ordered by citation count) include:

- Masur & Minsky, 1998 [MM98]: The authors use train tracks to describe the curve complex of a surface, i.e. a structure that contains information on possible configurations for sets of disjoint closed curves on a surface.

- Hamenstädt, 2005 [Ham06]: Hamenstädt uses train tracks to once-again describe the curve complex of a surface.

- Birman & Beendle, 2004, [BB05]: The authors give a survey on braid theory. They also mention train tracks as a possible solution to the conjugation problem for braids.

A somewhat more modern exposition is the preprint by Mosher [Mos03b]. The train track is carefully introduced and their structure is analysed in-depth.

In our research, we are mainly interested in invariant train tracks, i.e. train tracks that represent measured foliations. An algorithm for generating those train tracks is described by Bestvina and Handel [BH95]. The algorithm itself is better explained by Boyland [Boy94] and by [Iss12] but the paper has still been massively influential.

As for easily accessible literature on train tracks, there is the following. Each entry can easily be read by undergraduate students:

- Blog post by Johnson [Joh14]: Johnson defines train tracks on the torus and explains which curves are carried by the train tracks. This is then linked to Farey-intervals

- Article by Mosher [Mos03a]: Mosher explains Train tracks to a non-expert audience

- Master thesis by Issa [Iss12]: Issa explains measured foliations, laminations, train tracks and train track splittings sequences in simple terms

- Blog Post by Margalit [Mar23]: Margalit gives a short, intuitive bird's eye view on the topic of mapping class groups of surfaces and explains where train tracks fit into all of this.

- Braids and Dynamics by Thiffeault [Thi22]: The author gives a very accessible introduction to braid theory. Train tracks and the Bestvina-Handel-algorithm are introduced in the context of braids. The explanations are easy to follow.

2.3 Agol cycles

Splitting sequences have been studied for a long time. Penner and Harer [PH91] already described the splitting operation, alongside the shift and th fold operation. However, to my knowledge, Agol was the first to propose a natural splitting sequence, realized by splitting all branches with the maximal weights [Ago11]. Doing this, he was able to properly define the Agol cycle as a conjugacy-invariant of pseudo-Anosov maps. Agol's ideas have been inspired by work by Hamenstädt [Ham09]. As it stands, there are only a handful of papers, using his splitting sequence:

- Non-geometric veering triangulations [HIS16]: The three authors present an application which can be used to calculate triangulations of mapping class tori. As part of the functionality, the application can also output Agol cycle information.
- Agol cycles of pseudo-Anosov 3-braids [AK23]: This paper explicitly calculates Agol cycles for 3-braids.
- Complete description of Agol cycles of pseudo-Anosov 3-braids [KK23]: This paper defines train track on the torus and transfers them to the 3-punctured disc using a hyperbolic involution. The presented train tracks are described by Farey intervals, inspired by Mosher's work [Mos03b]

On the topic of splitting sequence, which predate Agol's paper, there are:

- On train-track splitting sequences [MMS12]: Has splitting sequence in the name
- Dilatations and Continued fraction [Bau92]: Uses folding sequences to get number-theoretical porperties from train tracks.

Since we mentioned the paper by Agol, we also have to mention a paper by Agol and Tsang [AT24]. This paper fixed an error which was introduced into [Ago11].

Chapter 3

Theory

In his paper [Ago11]) Agol states that periodic splitting sequences are a topological analogue of continued fractions, since they both seem to have similar properties. The aim of this report is to examine this connection for a simple case, the once-punctured torus and then for the twice-punctured torus. Due to time constraints, we won't be able to generalise the results to arbitrary surfaces $\S_{g,n}$. But simply stated, the generalisation works by identifying measured train tracks with p-adic continued fractions as introduced in [Bau92]. For a train track in an Agol cycle, the p-adic continued fraction becomes periodic and we obtain an explicit description of the train track weights, the Agol cycle and the dilatation of the map.

3.1 The correspondence between train tracks and continued fractions on the once-punctured torus

On the once-punctured torus it is possible to identify a special set of measured train tracks and continued fractions. In short, all train tracks on the once-punctured torus are homeomorphic to a train track consisting of two curves of slope 0 and ∞ where the crossing has been smoothed out into a large branch. We can give the branch a positive or negative slope, giving us two topological types. The ratio between the 0-branch weight and ∞-branch weight gives a (possibly infinite) real numbered slope whose continued fraction expansion determines the splitting sequence of the measured train track. This is mostly well-known theory but we will give a new proof by interpreting the continued fraction as a rectangle.

3.1.1 Pseudo-Anosov Homeomorphisms

Let $S = S_{1,1}$ be the once-punctured torus. [KK23] showed the following. We take the once-punctured torus to be a square with opposite edges identified and define two simple closed closed curves c_1, c_2 on top.

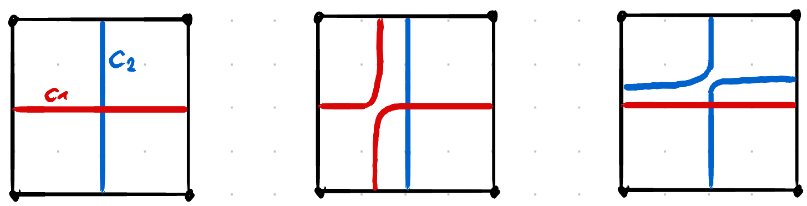


Figure 3.1: The maps f_L and f_R

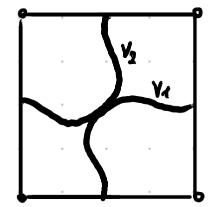


Figure 3.2: Train track $(\tau, \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right))$

We want to study the set of pseudo-Anosov homeomorphisms lying in the semigroup generated by the left-handed Dehn twist $f_L := \delta_{c_2}^{-1}$ around c_2 and the right-handed Dehn-Twist $f_R := \delta_{c_1}$ around c_1.

Definition 3.1.1 (Set maps generated by f_L, f_R)**.** The set of all maps generated by f_L, f_R is denoted by $F(f_L, f_R)$

From Penner's construction we get a necessary and sufficient condition for a map to be pseudo-Anosov.

Corollary 3.1.2. *For $l_i, r_i \in \mathbb{N}_0$ a map $f = f_L^{l_1} f_R^{r_1} ... f_L^{l_n} f_R^{r_n}$ is pseudo-Anosov if and only if each generator f_L, f_R occurs at least once in the product.*

Up to conjugacy all maps $f \in F(f_L, f_R)$ are described by products of the form $f = f_L^{l_1} f_R^{r_1} ... f_L^{l_n} f_R^{r_n}$ where $l_i, r_i \in \mathbb{N}$ and $n \in \mathbb{N}$.

Note that we are using a different notation from the one introduced in [KK23], which was $f = f_R^{r_n} f_L^{l_n} ... f_R^{r_1} f_L^{l_1}$. Our convention will later lead to nicer descriptions of the associated continued fractions.

3.1.2 Invariant train tracks

Definition 3.1.3. Due to the switch condition, the measure of the measured train track in Figure 3.2 is determined by two positive numbers $v_1, v_2 \in \mathbb{R}_{>0}$ and denoted by $(\tau, \mu) := (\tau, \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right))$.
The train track is said to have the **slope** $s := v_2/v_1$. The induced projective train track is denoted by $(\tau, [\mu]_p) := (\tau, [s]_p)$

Lemma 3.1.4. *Let $f = f_L^{l_1} f_R^{r_1} ... f_L^{l_n} f_R^{r_n}$ be a pseudo-Anosov map. By Penner's construction, the invariant train track $(\tau, v) = (\tau, \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right))$ is obtained by flattening the intersections of the curves. The incidence matrices of the generators f_R, f_L, are $R = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ respectively. The weights are therefore given by the Perron-Frobenius eigenvector v of the incidence matrix*

$$M = L^{l_1} R^{r_1} ... L^{l_n} R^{l_n}$$

We note that $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^a = \left(\begin{smallmatrix}0&1\\1&a\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) = D[a]\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^a = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\1&a\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)D[a]$ and transform M as follows:

$$\begin{aligned}
M &= L^{l_1}R^{r_1}...L^{l_n}R^{r_n}\\
&= \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^{l_1}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{r_1}...\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)^{l_n}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)^{r_n}\\
&= \left(D[l_1]\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)D[r_1]\right)...\left(D[l_n]\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)D[r_n]\right)\\
&= D[l_1]D[r_1]...D[l_n]D[r_n]
\end{aligned}$$

This allows us to apply theorem 1.2.3.

Corollary 3.1.5. *Let the assumptions be as in lemma 3.1.4. The slope s of the train track can be described by a continued fraction:*

$$s = [\overline{l_1, r_1, ..., l_n, r_n}]$$

The dilatation λ of f can be described explicitly as

$$\lambda = \prod_{v=1}^{p}(T^{v-1}[\overline{l_1, r_1, ..., l_n, r_n}])$$

3.1.3 Rectangle models

We established in corollary 3.1.5 that for a given homeomorphism $f = f_L^{l_1} f_R^{r_1} ... f_L^{l_n} f_R^{r_n}$, the slope of the above train track (τ, v) can be described by a continued fraction $[l_1, r_1, l_2, r_2, ...]$. This continued fraction can be represented by a rectangle partitioned into squares in such a way that $l_1, r_1, l_2, r_2, ...$ square are inserted alternatively from the bottom and the right. A maximal split on the train track changes the slope which induces an operation on the rectangle model. Next, we will show that a maximal split corresponds directly to the removal of a square as in figures 1.1 and 1.2.

Definition 3.1.6. For the train track (tau, v), the rectangle rect(v) is called the *equivalent rectangle to* (τ, v).

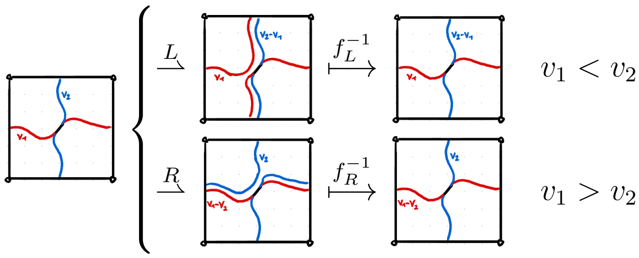
3.1.4 Agol cycles

The rectangle visualisation can be used to easily prove a result about splitting sequences of train tracks. All results are only valid for $v_1 \neq v_2$ but this won't be a problem later.

Proposition 3.1.7. *A maximal split on the train track (τ, v) has the following effect:*

$$(\tau, \left(\begin{smallmatrix}v_1\\v_2\end{smallmatrix}\right)) \begin{cases} \xrightarrow{L} \overset{f_L^{-1}}{\longmapsto} (\tau, \left(\begin{smallmatrix}v_1\\v_2-v_1\end{smallmatrix}\right)) & v_1 < v_2 \\ \xrightarrow{R} \overset{f_R^{-1}}{\longmapsto} (\tau, \left(\begin{smallmatrix}v_1-v_2\\v_2\end{smallmatrix}\right)) & v_1 > v_2 \end{cases}$$

Proof. We do the proof pictorially.



□

Theorem 3.1.8 ([KK23]). *For $i \in \mathbb{N}$ let $l_i, r_i \mathbb{N}$. The splitting sequence of the projective train track $(\tau, [s]_p)$ can be described as follows:*

$$(f_L^{-1} \circ \xrightarrow{L})(\tau, [[l_1, r_1, ...]]_p) = (\tau, [[l_1 - 1, r_1, ...]]_p)$$
$$(f_R^{-1} \circ \xrightarrow{R})(\tau, [[0, r_1, l_2, ...]]_p) = (\tau, [[0, r_1 - 1, l_2, ...]]_p)$$

Consequently:

$$(f_L^{-1} \circ \xrightarrow{L})^{l_1}(\tau, [[l_1, r_1, l_2, ...]]_p) = (\tau, [0, r_1, l_2, ...])$$
$$(f_R^{-1} \circ \xrightarrow{R})^{r_1}(\tau, [[0, r_1, l_2, ...]]_p) = (\tau, [[l_2, r_2 ...]]_p)$$

Proof. It is known from proposition 3.1.7 how a train track behaves under a maximal split. To show the above for a projective train track $(\tau, [s]_p)$, we will choose a representative train track (τ, v). The maximal split induces an operation on the equivalent rectangle $\mathrm{rect}(v)$. We will then use the connection between rectangles and continued fractions to show the theorem.

- If the slope of the measure is $s = [l_1, r_1, ...]$, then $v_1 < v_2$. The maximal splitting transforms the equivalent rectangle into $\mathrm{rect}(\,{}^{v_1}_{v_2 - v_1})$, i.e. removes a bottom square. The new rectangle then admits the partition $[l_1 - 1, r_1, l_2, r_2, ...]$.

- If the slope of the measure is $s = [0, r_1, ...]$, then $v_1 > v_2$. The maximal splitting transforms the equivalent rectangle into $\mathrm{rect}(\,{}^{v_1 - v_2}_{v_2})$, i.e. removes a right square. The new rectangle then admits the partition $[0, r_1 - 0, l_2, r_2, ...]$. For the final step, we remind ourselves that $[0, 0, l_2, r_2, l_3, r_3, ...] = [l_2, r_2, l_3, r_3, ...]$.

□

Corollary 3.1.9. *Let $s = [l_1, r_1, l_2, r_2, ...]$ be a infinite continued fraction and $l_i, r_i \geq 1$. The splitting sequence of the projective train track $(\tau, [s]_p)$ can be described as follows:*

$$(\tau, [[l_1, r_1, l_2, r_2, ...]]_p) \xrightarrow{L}{}^{l_1} \xrightarrow{R}{}^{r_1} \xmapsto{f_L^{-l_1}} \xmapsto{f_R^{-r_1}} (\tau, [[l_2, r_2, l_3, r_3, ...]]_p)$$

Theorem 3.1.10. *Let $f = f_L^{l_n} f_R^{r_n} ... f_L^{l_1} f_R^{r_1} \in \mathcal{L}$ be a pseudo-Anosov map. Let $v \in \mathbb{R}_+^2$ be a vector with slope $s = [\overline{l_1, r_1, ..., l_n, r_n}]$. Then the measured train track (τ, v) is part of the Agol cycle of f. The Agol cycle beginning with (τ, v) is:*

$$(\tau, v) \overset{L}{\rightharpoonup}^{l_1} \overset{R}{\rightharpoonup}^{r_1} \dots \overset{L}{\rightharpoonup}^{l_n} \overset{R}{\rightharpoonup}^{r_n} f(\tau, \lambda^{-1} v)$$

Furthermore, the length of the Agol cycle is $\sum l_i + r_i$

Proof. We apply corollary 3.1.9 n times to the train track (τ, v). Bacause the slope s of v is periodic, the train track will become $(\tau, c^{-1}v)$ (for a $c \in \mathbb{R}_+$). By theorem 1.5.16 this already proves that the above is an Agol cycle and that c ust have been equal to the dilatation λ. ☐

3.2 The correspondence on a familiy of homeomorphisms of the twice-punctured torus

3.2.1 Pseudo-Anosov Homeomorphisms

We apply the theory from above to the twice-punctured torus. As in the previous case, and as seen in 3.3 we define simple closed curves, labelled $1, 2, 3$ on the twice-punctured torus.

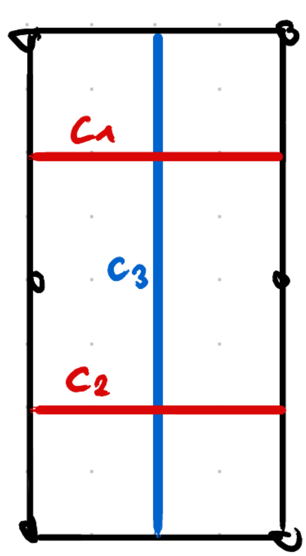


Figure 3.3: Simple closed curves c_1, c_2, c_3 on the twice-punctured torus

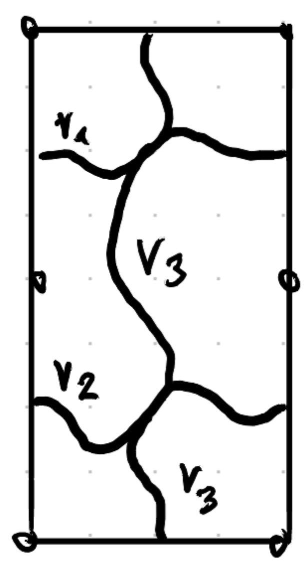


Figure 3.4: Measured train track on the twice-punctured torus

This time we will study maps composed of right-handed Dehn-twists around the curves c_1, c_2 and left-handed Dehn-twists around the curve c_3. We call those maps $f_R := \delta_{c_1}^{-1}, f_{R'} := \delta_{c_2}^{-1}$ and $f_L := \delta_{c_3}^{-1}$ respectively. As before, we define $F(f_R, f_{R'}, f_L)$.

Definition 3.2.1 (Set maps generated by $f_L,\ f_R, f_R'$)**.** The set of all maps generated by $f_R, f_L, f_{R'}$ is denoted by $(f_R, f_{R'}, f_L)$

From Penner's construction we again get a condition for when maps are pseudo-Anosov:

Corollary 3.2.2. *For $l_i, r_i, r_i' \in \mathbb{N}_0$ a map $f = f_L^{l_1} f_R^{r_1} f_{R'}^{r_1'} ... f_L^{l_n} f_R^{r_n} f_{R'}^{r_n'}$ is pseudo-Anosov if and only if each generator $f_R, f_{R'}, f_L$ occurs at least once in the product.*

Since we are interested in maps up to conjugacy, we will study maps of the form $f = f_L^{l_1} f_R^{r_1} f_{R'}^{r_1'} ... f_L^{l_n} f_R^{r_n} f_{R'}^{r_n'}$ where $l_i \geq 1$ and $r_i + r_i' \geq 1$.

3.2.2 Invariant train tracks

Definition 3.2.3. Due to the switch condition, the measure of the measured train track in Figure 3.4 is determined by three positive numbers $v_1, v_2, v_3 \in \mathbb{R}_{>0}$ and denoted by $(\tau, \mu) := (\tau, \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right))$.

The train track is said to have the **slope** $s := \begin{pmatrix} v_2/v_1 \\ v_3/v_1 \end{pmatrix}$. The induced projective train track is denoted by $(\tau, [\mu]_p) := (\tau, [s]_p)$

Lemma 3.2.4. *Let $f = f_L^{l_1} f_R^{r_1} f_{R'}^{r_1'} ... f_L^{l_n} f_R^{r_n} f_{R'}^{r_n'}$ be a pseudo-Anosov map. By Penner's construction, the invariant train track $(\tau, v) = (\tau, \left(\begin{smallmatrix} v_1 \\ v_2 \\ v_3 \end{smallmatrix}\right))$ is obtained by flattening the intersections of the curves. The incidence matrices of the generators $f_R, f_{R'}, f_L$, are $R = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) R' = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ respectively. The weights are therefore given by the Perron-Frobenius eigenvector v of the incidence matrix*

$$M = L^{l_1} R^{r_1} R'^{r_1'} ... L^{l_n} R^{r_n} R'^{r_n'}$$

As previously, we transform the matrices R, R', L into the $D_2[a]$-matrices. This can be done by applying elementary operations to our matrix. The detailed approach is explained in [Bau92].

Lemma 3.2.5. *The matrices R, R', L can be written as follows:*

- $L^a = D_2[\left(\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right)] D_2[\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)] D_2[0] = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & a \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$

- $R^a = D_2[0]^2 D_2[\left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}\right)] = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)^2 \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & a \\ 0 & 1 & 0 \end{smallmatrix}\right)$

- $R'^a = D_2[0]^2 D_2[\left(\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right)] = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)^2 \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & a \end{smallmatrix}\right)$

- $L^l R^r R'^{r'} = D_2[\binom{0}{l}] D_2[\binom{l}{0}] D_2[\binom{r}{r'}]$

Proof. Calculation. □

Again we apply theorem 1.2.3. From this we can describe the weights of the invariant measured train track as a 2-adic continued fraction.

Corollary 3.2.6. *Let the assumptions be as in Lemma 3.2.4. The slope s of the train track can be described by a continued fraction:*

$$s = \left[\overline{\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, ..., \binom{0}{l_n}, \binom{l_n}{0}, \binom{r_n'}{r_n}}\right]$$

The dilatation λ of f can be described explicitly as

$$\begin{aligned}\lambda &= \prod_{v=1}^{n} \left(T^{v-1} \left[\overline{\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, ..., \binom{0}{l_n}, \binom{l_n}{0}, \binom{r_n'}{r_n}}\right]\right)|_2 \\ &= \prod_{v=1}^{n} \left(T^{v-1} \left[\overline{l_1, r_1 + \ r_1', ..., l_n, r_n' + r_n}\right]\right)\end{aligned}$$

Proof. The formulas for s and $\lambda = \prod_{v=1}^{n} \left(T^{v-1} \left[\overline{\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, ..., \binom{0}{l_n}, \binom{l_n}{0}, \binom{r_n'}{r_n}}\right]\right)|_2$ follow directly from 1.2.3.
We will prove the equality by analysing how the matrix $L^l R^r R^{r'}$ act on the last component of a vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$:

$$\begin{aligned} L^l R^r R^{r'} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}|_3 &= D_2[\binom{0}{l}] D_2[\binom{l}{0}] D_2[\binom{r}{r'}] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + r v_3 \\ v_2 + r' v_3 \\ v_3 + l(v_1 + r v_3) + l(v_2 + r' v_3) \end{pmatrix}|_3 \\ &= v_3 + l((v_1 + v_2) + (r + r')v_3) = \begin{pmatrix} (v_1+v_2)+(r+r')v_3 \\ v_3 + l(v_1+v_2+(r+r')v_3) \end{pmatrix}|_2 \\ &= D_1[l] D_1[r + r'] \binom{v_1+v_2}{v_3}|_2 \end{aligned}$$

If v is the Perron-Frobenius eigenvector $M = L^{l_1} R^{r_1} R'^{r_1'} ... L^{l_n} R^{r_n} R'^{r_n'}$ then it has slope s by theorem 1.2.3 and eigenvalue λ. The previous calculation gives us:

$$\begin{aligned} \lambda v_3 = M \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}|_3 &= D_2[\binom{0}{l_1}] D_2[\binom{l_1}{0}] D_2[\binom{r_n}{r_n'}] ... D_2[\binom{0}{l_n}] D_2[\binom{l_n}{0}] D_2[\binom{r_n}{r_n'}] \binom{v_1+v_2}{v_3}|_3 \\ &= D_1[l_1] D_1[r_1 + r_1'] ... D_1[l_n] D_1[r_n + r_n'] \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}|_2 \\ &= \prod_{v=1}^{n} \left(T^{v-1} \left[\overline{l_1, r_1 + \ r_1', ..., l_n, r_n' + r_n}\right]\right) \end{aligned}$$

by the same theorem. □

3.2.3 Rectangle Models

Just as in the once-punctured case, we want to introduce a rectangle model to visualise the train track weights. Since the train track introduced on the twice puncture is defined by three parameters it seems natural to use a cuboid with the side lengths representing the train track weights. However, it turns out that for out family of homeomorphisms we can use as split rectangle as well.

In the following we will define the split rectangle in a way analogous to a family of 2-ary continued fractions.

Definition 3.2.7 (Split rectangle)**.** Let $v_1, v_2, v_3 \in \mathbb{R}_+$. The split rectangle with side lengths as in Figure 3.5 is denoted $\operatorname{rect}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

Definition 3.2.8 (Square partition)**.** Let $\operatorname{rect}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be a rectangle. We say, it *admits a square partition* $[q_1, p_1, p_1', ...]$ if it can be partitioned into a finite amount of squares as seen in the picture 3.6.

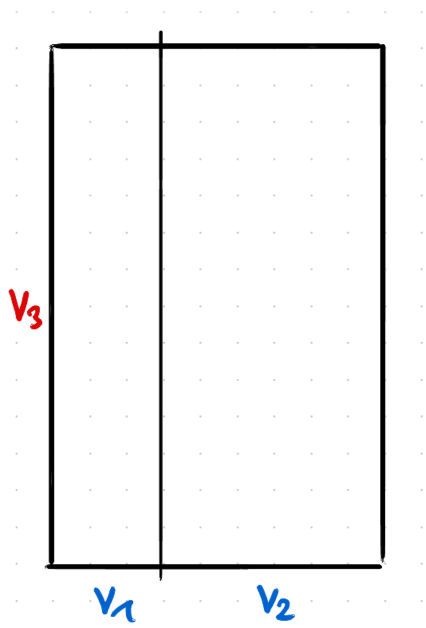


Figure 3.5: Split rectangle

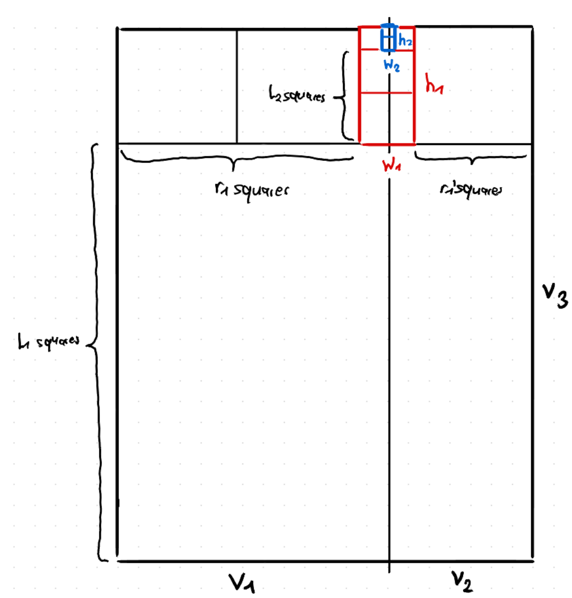


Figure 3.6: Rectangle admitting a partition

Proposition 3.2.9. *Let* $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3_+$ *with slope* $s = \begin{pmatrix} v_2/v_1 \\ v_3/v_1 \end{pmatrix}$. $rect_+(v)$ *admits the square partition* $[l_1, r_1, r_1', l_2, r_2, r_2', ...]$ *if and only if s has the* 2*-ary continued fraction expansion* $s = \left[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r_2'}, ...\right]$

Proof. The infinite continued fraction is defined as the limit of a series of matrix products.

$$[w]_p = \lim_{t\to\infty} \left[D_2[\binom{0}{l_1}] D_2[\binom{l_1}{0}] D_2[\binom{r_1}{r_1'}] ... D_2[\binom{0}{l_n}] D_2[\binom{l_n}{0}] D_2[\binom{r_n}{r_n'}] \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right]_{\underline{\underline{p}}}$$

We note that removing the first block of matrices $D_2[\binom{0}{l_1}]D_2[\binom{l_1}{0}]D_2[\binom{r_1}{r_1'}]$ has the following effect on a vector

$$v' := D_2[\binom{r_1}{r_1'}]^{-1}D_2[\binom{l_1}{0}]^{-1}D_2[\binom{0}{l_1}]^{-1}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1-r_1(v_3-l_1(v_1+v_2)) \\ v_2-r_1'(v_3-l_1(v_1+v_2)) \\ (v_3-l_1(v_1+v_2)) \end{pmatrix}$$

Similarly, removing the highlighted squares from the rectangle has the same effect, giving us the rectangle

$$\operatorname{rect}\begin{pmatrix} v_1-r_1(v_3-l_1(v_1+v_2)) \\ v_2-r_1'(v_3-l_1(v_1+v_2)) \\ (v_3-l_1(v_1+v_2)) \end{pmatrix} = \operatorname{rect}\left(D_2[\binom{r_1}{r_1'}]^{-1}D_2[\binom{l_1}{0}]^{-1}D_2[\binom{0}{l_1}]^{-1}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \operatorname{rect}(v')$$

We now proove the equivalence. By proposition 1.2.2 s has the continued fraction expansion $\left[\binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r_2'}, \binom{0}{l_3}, \binom{l_3}{0}, \binom{r_3}{r_3'}, ...\right]$ if and only if

$$\lim_{n\to\infty} \left(D_2[\binom{r_n}{r_n'}]^{-1}D_2[\binom{l_n}{0}]^{-1}D_2[\binom{0}{l_n}]^{-1}\right)...\left(D_2[\binom{r_1}{r_1'}]^{-1}D_2[\binom{l_1}{0}]^{-1}D_2[\binom{0}{l_1}]^{-1}\right)\binom{1}{s} = 0$$

which, due to the previous analogy, holds if and only if $\operatorname{rect}(v)$ admits the square partition $[l_1, r_1, r_1', l_2, r_2, r_2', ...]$. □

Lemma 3.2.10. *For $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, let rect$(v)$ be a rectangle which admits a square partition $(q_1, p_1, p_1', q_2, p_2, p_2', ...)$, where $q_i \geq 1$ for all $i \in \mathbb{N}$. Then we have*

$$\begin{aligned} v_1 < v_2 &\iff (p_1, p_2, ...) \prec (p_1', p_2', ...) \\ v_1 = v_2 &\iff (p_1, p_2, ...) = (p_1', p_2', ...) \\ v_1 > v_2 &\iff (p_1, p_2, ...) \succ (p_1', p_2', ...) \end{aligned}$$

where $\prec, =, \succ$ compare the sequences lexicographically.

Proof. We draw the partitioned rectangle, similarly to figure 3.6. Let w_i, h_i denote the widths and heights of the rectangles as in the picture. This lets us describe v_1, v_2 with the following sums:

$$v_1 = \sum_{i=1}^{\infty} p_i h_{i+1} \text{ and } v_2 = \sum_{i=1}^{\infty} p_i' h_{i+1}$$

The fact $\sum_{j=i}^{\infty} p_j h_{j+1}, \sum_{j=i}^{\infty} p_j' h_{j+1} \leq w_i < h_i$ also clearly follows from the picture. The first inequality is obviously true and since $q_i > 0$ all highlighted rectangles in fig 3.6 are higher than wide, i.e. $w_i < h_i$.
If $(p_1, p_2, ...) \prec (p_1', p_2', ...)$, then there is a smallest i such that $p_i < p_i'$, with all previous coefficients being equal. We then have

$$p_i h_{i+1} < p_i' h_{i+1} \iff p_i h_{i+1} + \sum_{j=i+1}^{\infty} p_j h_{j+1} < p_i' h_{i+1} + \sum_{j=i+1}^{\infty} p_j' h_{j+1} \iff \sum_{j=1}^{\infty} p_j h_{j+1} < \sum_{j=1}^{\infty} p_j' h_{j+1}$$

□

We have just identified continued fraction expansions with square partitions of rectangles. Next we will identify rectangles with train tracks and later use the fact that a maximal split corresponds to the removal of a square from a rectangle.

Definition 3.2.11 (Equivalent rectangle)**.** Let $v \in \mathbb{R}^n_+$. For the train track (τ, v), the split rectangle $\mathrm{rect}(v)$ is called the *equivalent rectangle to* (τ, v).

3.2.4 Agol cycle

The modified rectangle model will be used to give an intuitive and convincing proof for the Agol cycle of a map f. But first we calculate how the train track changes under a maximal split. We will notice some differences, depending on whether $v_1 = v_2$ or $v_1 \neq v_2$. Later these differences in weight will correspond to a difference in a r_i, r_i' in a map $f = f_L^{l_1} f_R^{r_1} f_{R'}^{r_1'} \ldots f_L^{l_n} f_R^{r_n} f_{R'}^{r_n'}$. Maps for which $r_i = r_i'$ for all $i \in \{1, \ldots, n\}$ will be called *symmetric*. If there is an i s.t. $r_i \neq r_i'$ then f is called asymmetric. Generally speaking, in symmetric maps maximal splits at multiple branches occure more often, making Agol cycles shorter than in the asymmetric case.

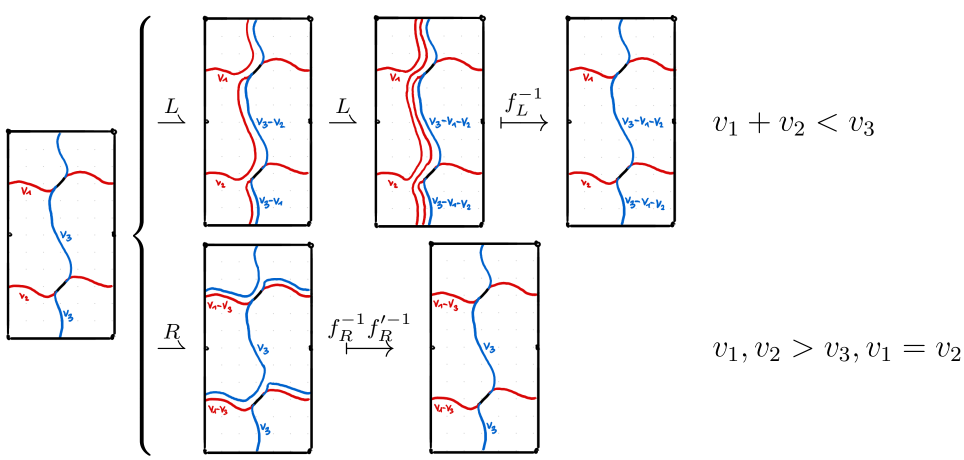
Proposition 3.2.12. *A maximal split on the train track* (τ, v) *has the following effect: For the symmetric case, i.e.* $v_1 = v_2$*, we have:*

$$\left(\tau, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) \begin{cases} \overset{L}{\rightharpoondown}{}^2 \overset{f_L^{-1}}{\longmapsto} \left(\tau, \begin{pmatrix} v_1 \\ v_2 \\ v_3 - v_1 - v_2 \end{pmatrix}\right) & v_1 + v_2 < v_3 \\ \overset{R}{\rightharpoondown} \overset{f_R^{-1} f_R'^{-1}}{\longmapsto} \left(\tau, \begin{pmatrix} v_1 - v_3 \\ v_2 - v_3 \\ v_3 \end{pmatrix}\right) & v_1, v_2 > v_3, v_1 = v_2 \end{cases}$$

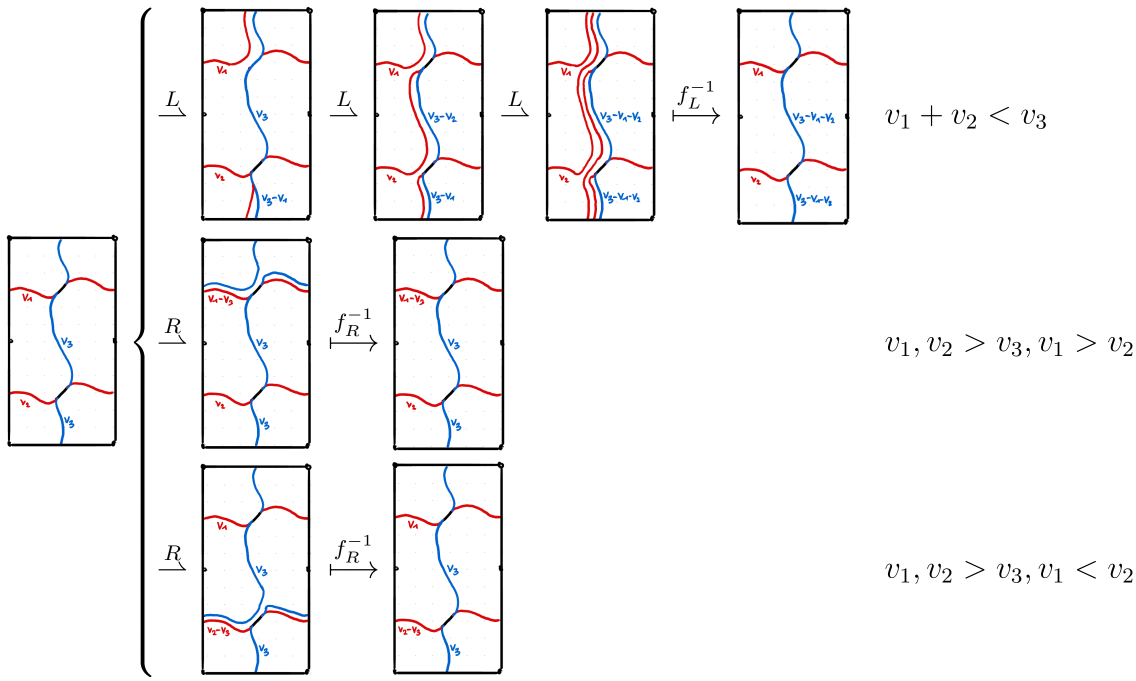
For the asymmetric case $v_1 \neq v_2$*, we have:*

$$\left(\tau, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) \begin{cases} \overset{L}{\rightharpoondown}{}^3 \overset{f_L^{-1}}{\longmapsto} \left(\tau, \begin{pmatrix} v_1 \\ v_2 \\ v_3 - v_1 - v_2 \end{pmatrix}\right) & v_1 + v_2 < v_3 \\ \overset{R}{\rightharpoondown} \overset{f_R^{-1}}{\longmapsto} \left(\tau, \begin{pmatrix} v_1 - v_3 \\ v_2 \\ v_3 \end{pmatrix}\right) & v_1, v_2 > v_3, v_1 > v_2 \\ \overset{R}{\rightharpoondown} \overset{f_R^{-1}}{\longmapsto} \left(\tau, \begin{pmatrix} v_1 \\ v_2 - v_3 \\ v_3 \end{pmatrix}\right) & v_1, v_2 > v_3, v_1 < v_2 \end{cases}$$

Proof. We do the proof pictorially. For the symmetric case, i.e. $v_1 = v_2$, we have:



For the asymmetric case $v_1 \neq v_2$ (w.l.o.g. $v_1 < v_2$), we have:



□

Theorem 3.2.13. *Let* $s = [\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r_2'}, \ldots]$ *be a 2-ary infinite continued fraction with the implied pattern and* $l_i \geq 1, r_i + r_i' \geq 1$ *and possibly* $l_1 = 0$. *The splitting sequence of the projective train track* $(\tau, [s]_p)$ *can be described as follows:*

- *In the symmetric case* $(r_i = r_i')$

$$(f_L^{-1} \circ \xrightarrow{L}^2)(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[\binom{0}{l_1-1}, \binom{l_1-1}{0}, \binom{r_1}{r_1'}, , \ldots]]_p)$$
$$(f_R^{-1} f_{R'}^{-1} \circ \xrightarrow{L})(\tau, [[0, 0, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[0, 0, \binom{r_1-1}{r_1'-1}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p)$$

Consequently:

$$(f_L^{-1} \circ \xrightarrow{L}^2)^{l_1}(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[0, 0, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p)$$
$$(f_R^{-1} f_{R'}^{-1} \circ \xrightarrow{R})^{r_1}(\tau, [[0, 0, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[\binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p)$$

- *In the asymmetric case* $(r_i \neq r_i'$ *for one* $r_i)$

$$(f_L^{-1} \circ \xrightarrow{L}^3)(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[\binom{0}{l_1-1}, \binom{l_1-1}{0}, \binom{r_1}{r_1'}, , \ldots]]_p)$$
$$(f_R^{-1} \circ \xrightarrow{R})(\tau, [[0, 0, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[0, 0, \binom{r_1-1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p)$$
$$(f_{R'}^{-1} \circ \xrightarrow{R})(\tau, [[0, 0, \binom{r_1}{r_1'}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p) = (\tau, [[0, 0, \binom{r_1}{r_1'-1}, \binom{0}{l_2}, \binom{l_2}{0}, \ldots]]_p)$$

Consequently:

$$(f_L^{-1}\circ \overset{L}{\rightharpoonup}{}^{3})^{l_1}(\tau,[[\tbinom{0}{l_1},\tbinom{l_1}{0},\tbinom{r_1}{r'_1},\tbinom{0}{l_2},\tbinom{l_2}{0},...]]_p)=(\tau,[[0,0,\tbinom{r_1}{r'_1},\tbinom{0}{l_2},\tbinom{l_2}{0},...]]_p)$$
$$(f_R^{-r_1}f_{R'}^{-r'_1}\circ \overset{R}{\rightharpoonup}{}^{r_1+r'_1})(\tau,[[0,0,\tbinom{r_1}{r'_1},\tbinom{0}{l_2},\tbinom{l_2}{0},...]]_p)=(\tau,[[\tbinom{0}{l_2},\tbinom{l_2}{0},...]]_p)$$

Proof. It is known from proposition 3.2.12 how a train track behaves under a maximal split. To show the above for a projective train track $(\tau,[s]_p)$, we choose a representative train track (τ,v). The maximal split induces an operation on the equivalent rectangle $\mathrm{rect}(v)$. We will then use the connection between split rectangles and continued fractions to show the theorem.

- Symmetric train tracks ($v_1=v_2$): The train track can change as follows

$$(\tau,\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix})\begin{cases}\overset{L}{\rightharpoonup}{}^{2}\xmapsto{f_L^{-1}}(\tau,\begin{pmatrix}v_1\\v_2\\v_3-v_1-v_2\end{pmatrix}) & v_1+v_2<v_3\\ \overset{R}{\rightharpoonup}\xmapsto{f_R^{-1}f_R'^{-1}}(\tau,\begin{pmatrix}v_1-v_3\\v_2-v_3\\v_3\end{pmatrix}) & v_1,v_2>v_3, v_1=v_2\end{cases}$$

  A representative of the train track $(\tau,[[\binom{0}{l_1},\binom{l_1}{0},\binom{r_1}{r'_1},\binom{0}{l_2},\binom{l_2}{0},...]]_p)$ fulfills the conditions of the first case. The maximal splitting removes the bottom square of the equivalent rectangle. The new rectangle then admits the partition $[l_1-1,r_1,r'_1,l_2,r_2,r'_2,...]$. For the final step, we use the fact, that the 2-ary continued fraction has the property $[0,0,0,y^{(4)},y^{(5)},y^{(6)},...]=[y^{(4)},y^{(5)},y^{(6)},...]$.
  A representative of the train track $(\tau,[[0,0,\binom{r_1}{r'_1},\binom{0}{l_2},\binom{l_2}{0},...]]_p)$ fulfills the conditions of the second case. The maximal splitting removes the left and right square of the equivalent rectangle. The new rectangle then admits the partition $[0,r_1-1,r'_1-1,l_2,r_2,r'_2,...]$.

- Asymmetric train tracks ($v_1\neq v_2$): The train track can change as follows

$$(\tau,\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix})\begin{cases}\overset{L}{\rightharpoonup}{}^{3}\xmapsto{f_L^{-1}}(\tau,\begin{pmatrix}v_1\\v_2\\v_3-v_1-v_2\end{pmatrix}) & v_1+v_2<v_3\\ \overset{R}{\rightharpoonup}\xmapsto{f_R^{-1}}(\tau,\begin{pmatrix}v_1-v_3\\v_2\\v_3\end{pmatrix}) & v_1,v_2>v_3, v_1>v_2\\ \overset{R}{\rightharpoonup}\xmapsto{f_R^{-1}}(\tau,\begin{pmatrix}v_1\\v_2-v_3\\v_3\end{pmatrix}) & v_1,v_2>v_3, v_1<v_2\end{cases}$$

  A representative of the train track $(\tau,[[\binom{0}{l_1},\binom{l_1}{0},\binom{r_1}{r'_1},\binom{0}{l_2},\binom{l_2}{0},...]]_p)$ fulfills the conditions of the first case. The maximal splitting removes the bottom square of the equivalent rectangle. The new rectangle then admits the partition $[l_1-1,r_1,r'_1,l_2,r_2,r'_2,...]$.
  If $v_1>v_2$ (resp. $v_1<v_2$) then a representative of the train track $(\tau,[[\binom{0}{l_1},\binom{l_1}{0},\binom{r_1}{r'_1},\binom{0}{l_2},\binom{l_2}{0},...]]_p)$ fulfills the conditions of the second (resp. third) case. The maximal splitting removes the left (resp. right) square of the equivalent rectangle. The new rectangle then admits the partition $[0,r_1-1,r_1,l_2,r_2,r'_2,...]$ (resp. $[0,r_1-1,r'_1,l_2,r_2,r'_2,...]$).

□

Corollary 3.2.14. *Let* $s = [\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, \binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r'_2}, ...]$ *be a 2-ary infinite continued fraction with the implied pattern and* $l_i \geq 1, r_i + r'_i \geq 1$*. The splitting sequence of the projective train track* $(\tau, [s]_p)$ *can be described as follows:*

- *In the symmetric case* $(r_i = r'_i)$

$$(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, ...]]_p) \overset{L}{\rightharpoondown}{}^{2l_1} \overset{R}{\rightharpoondown}{}^{r_1} \overset{f_L^{-l_1}}{\longmapsto} \overset{f_R^{-r_1} f_{R'}^{-r'_1}}{\longmapsto} (\tau, [[\binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r'_2}, , ...]]_p)$$

- *In the asymmetric case* $(r_i \neq r'_i$ *for one* $r_i)$

$$(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, ...]]_p) \overset{L}{\rightharpoondown}{}^{3l_1} \overset{R}{\rightharpoondown}{}^{r_1 + r'_1} \overset{f_L^{-l_1}}{\longmapsto} \overset{f_R^{-r_1} f_{R'}^{-r'_1}}{\longmapsto} (\tau, [[\binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r'_2}, , ...]]_p)$$

Proof. The corollary directly follows from application of the theorem above:
In the symmetric case $(r_i = r'_i)$

$$\begin{aligned}(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, ...]]_p) &\overset{L}{\rightharpoondown}{}^{2l_1} \overset{f_L^{-l_1}}{\longmapsto} (\tau, [[\binom{0}{0}, \binom{0}{0}, \binom{r_2}{r'_2}, , ...]]_p) \\ &\overset{R}{\rightharpoondown}{}^{r_1} \overset{f_R^{-r_1} f_{R'}^{-r'_1}}{\longmapsto} (\tau, [[\binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r'_2}, ...]]_p)\end{aligned}$$

In the asymmetric case $(r_i \neq r'_i$ for one $r_i)$

$$\begin{aligned}(\tau, [[\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, ...]]_p) &\overset{L}{\rightharpoondown}{}^{3l_1} \overset{f_L^{-l_1}}{\longmapsto} (\tau, [[\binom{0}{0}, \binom{0}{0}, \binom{r_2}{r'_2}, , ...]]_p) \\ &\overset{R}{\rightharpoondown}{}^{r_1 + r'_1} \overset{f_R^{-r_1} f_{R'}^{-r'_1}}{\longmapsto} (\tau, [[\binom{0}{l_2}, \binom{l_2}{0}, \binom{r_2}{r'_2}, ...]]_p)\end{aligned}$$

□

Theorem 3.2.15. *Let* $f = \underline{f_L^{l_1} f_R^{r_1} f_{R'}^{r'_1} ... f_L^{l_n} f_R^{r_n} f_{R'}^{r'_n}} \in \mathcal{L}$ *be a pseudo-Anosov map. Let* $v \in \mathbb{R}^3_+$ *be a vector with slope* $s = [\overline{\binom{0}{l_1}, \binom{l_1}{0}, \binom{r_1}{r'_1}, ..., \binom{0}{l_n}, \binom{l_n}{0}, \binom{r_n}{r'_n}}]$*. Then the measured train track* (τ, v) *is part of the Agol cycle of* f*.*

- *If* f *is symmetric, i.e.* $r_i = r_i$ *for all* $i \in \{1, ..., n\}$*, then the Agol cycle beginning with* (τ, v) *is:*

$$(\tau, v) \overset{L}{\rightharpoondown}{}^{2l_1} \overset{R}{\rightharpoondown}{}^{r_1} \ldots \overset{L}{\rightharpoondown}{}^{2l_n} \overset{R}{\rightharpoondown}{}^{r_n} f(\tau, \lambda^{-1} v)$$

  Furthermore, the length of the Agol cycle is $\sum_{i=1}^n 2q_i + p_i$*.*

- *If* f *is assymmetric, i.e.* $r_i \neq r_i$ *for one* $i \in \{1, ..., n\}$*, then the Agol cycle beginning with* (τ, v) *is:*

$$(\tau, v) \overset{L}{\rightharpoondown}{}^{3l_1} \overset{R}{\rightharpoondown}{}^{r_1 + r'_1} \ldots \overset{L}{\rightharpoondown}{}^{3l_n} \overset{R}{\rightharpoondown}{}^{r_n + r'_n} f(\tau, \lambda^{-1} v)$$

  Furthermore, the length of the Agol cycle is $\sum_{i=1}^n 3q_i + p_i + p'_i$

Proof. We apply corollary 3.2.14 n times to the train track (τ, v). Since the slope s of v is periodic, the train track will eventually become $(\tau, c^{-1}v)$ (for a $c \in \mathbb{R}_+$). By **??** this already proves that the above is an Agol cycle and that c must have been equal to the dilatation λ. $\square$

Theorem 3.2.16. *Let* $f = f_L^{l_1} f_R^{r_1} f_{R'}^{r'_1} \ldots f_L^{l_n} f_R^{r_n} f_{R'}^{r'_n} \in \mathcal{L}$ *be a pseudo-Anosov map. Then the total splitting number of the Agol cycle with respect to* f *does not depend on symmetry/asymmetry. It is calculated by:*

$$N(f) = \sum_{i=1}^{n} 4q_i + p_i + p'_i$$

Proof. The proof will not be given here. The result can easily be verified in analogy to the Agol cycle length. $\square$

Chapter 4

Discussion

Many things have been left out. For one thing, the background could be much more extensive, to explain the topic to a graduate student just starting out. The obvious gaps are the lack of explanations for geodesic laminations, mapping class groups, the Thurston-Nielsen classification, pseudo-Anosov maps and probably some more.

The literature review is not exhaustive. I spent a large part of the year trying to figure out what exactly to research and therefore didn't have much time to survey the literature. For many mentioned of the articles mentioned I was only able to read the introduction and not much more. I would have liked to have made a mind map, showing how the different papers influenced each other, giving a useful overview of the subject. This would be useful, especially for me. The mind map might also highlight a suspicion of mine, namely that the combinatorial study of train tracks has split into two largely independent branches, one pioneered by Series et al. and the other by Agol et al. Integrating these two branches might yield fruitful results.

On the theory sinde, there were a lot of things I wanted to say but couldn't due to time constraints. I would have liked to explain the train tracks on the 3-punctured disc and the train tracks on the 4-punctured disc. It turns out that they are related to the train tracks of the once and twice-punctured torus by a (branched) double cover (see [KK23] for an example). There should be a construction to create the Agol cycle of any punctured disc from the Agol cycle of its double cover. At least for maps of Penner's construction (which descend to simple maps on the punctured disc)

Furthermore, the obvious connection between train tracks and laminations has not been well-discussed, even though this connection is very well known in the literature, having already been described by Penner in 1991 [PH91]. Nevertheless, there is an important connection to be made. If we have a train track with weights given by an n-ary continued fraction, then this train track carries a geodesic lamination with a specific slope. I suspect, this slope is given by the same continued fraction. The paper by Nikolaev [Nik03] might be extremely useful to further pursue this topic. In the case of the once-punctured torus, there is some theory on the cutting sequence of a geodesic, i.e. how a geodesic cuts through the tessellation, induced by universial cover of a surface. Work by Series [Ser85] suggests that this cutting sequence determines the left/right splits of a splitting sequence. But this is only a hunch and still very speculative.

Besides that, we didn't even finish the obvious next task. We have described the train tracks for a family of pseudo-Anosov homeomorphisms on the once-punctured and twice-punctured torus by 1 and 2-ary fractions. I suspect that it should be relatively easy to extend the construction to any map given by Penner's construction. However, when analysing thsse maps, it might be wise not to use the continued fractions defined by $D_n[y]$ but use the transpose $D_n[y]^t$ instead. The difference is subtle but it might better highlight the number-theoretical importance of train track splitting sequences. Intuitively, $D_n[y]$ describes to which weights a weight w_i should be added, whereas $D_n[y]^t$ describes the opposite perspective. Which weights should be added to w_i. I think this better reflects the nature of train track splitting sequences.

Finally, I wanted to generalise a theorem in Bauer. In short, [Bau92] contains a theorem which only works if there is a "folding sequence" connecting τ and $f(\tau)$. Bauer could not show this for any pseudo-Anosov map. But since an Agol cycle would give us such a "folding sequence", extending the the proof should not be too hard, there was just not enough time to write it out.

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